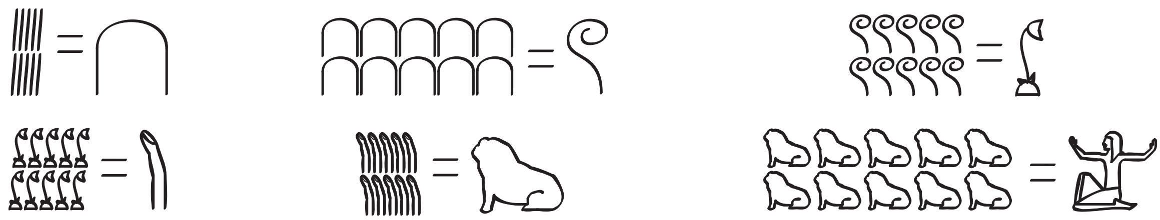


In the binary system we count on our fists instead of on our fingers. – Anonymous

CHAPTER 8 Base Numbers

8.1 Introduction

This chapter revolves around different methods for writing numbers. This may sound a little strange at first, but the way in which we usually write numbers (using ten digits) is simply a matter of convenience.

Why do we use ten digits? How would numbers look if we counted using a different number of digits? Would writing numbers using different digits make solving any problems easier? This chapter will help answer these questions.

8.2 Counting in Bundles

You already know about the decimal system. The decimal system is our method of writing numbers based on counting in groups of 10. In fact, *deci* is the Latin root for 10. A **numeral** is a representation of a number and each integer has its own numeral in the decimal system. For instance, the decimal numeral for the number "four" is 4, the decimal numeral for "seven" is 7, and the decimal numeral for "one hundred sixty-three" is 163.

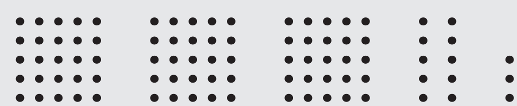
The system of counting in tens became popular because most people have ten fingers with which to count. If humans typically had 8 fingers (like Homer Simpson) then we would almost certainly be counting by 8's!

If right now you are asking, "How would counting by 8's work?"—don't worry! We will get to that soon enough.

Extra! *Most people would sooner die than think; in fact, they do so.* – Bertrand Russell

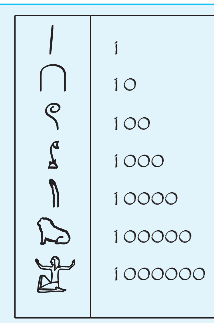
Problems

Problem 8.1:



Which of these three groups of dots is easiest to count? How many dots are in each picture?

Sidenote: Thousands of years ago in ancient Egypt, a number system was developed that shares similarity with our modern decimal system. Symbols were used to represent the smallest few integer powers of 10 as we see at right. Like other hieroglyphics they were written from right to left, biggest to smallest, and top to bottom. In order to make them easier to read, they are often written from left to right as below.



Positive integers were written using combinations of these symbols:

Problem 8.1: Which of the groups of dots on the previous page is easiest to count? How many dots are in each picture?

Solution for Problem 8.1: The numbers of dots in each picture are exactly the same. The dots in the third picture are arranged in groups that make them easier to count. The number of dots in each square group is $5^2 = 25$ and the number of dots in each row is 5. There are 3 square groups, 2 column groups, and 3 leftover dots for a total of

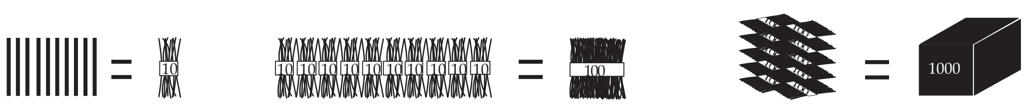
$$3 \cdot 5^2 + 2 \cdot 5 + 3 = 75 + 10 + 3 = 88.$$

□

Concept: It's easier to count a large number of objects when we group those objects as we count them.

In Problem 8.1, the second and third pictures group dots in ways that make them easier and easier to count. From the first picture to the second, complete groups of 5 dots are bundled together into strips of 5. Then, the third picture bundles strips of 5 into as many complete 5×5 squares as possible. Each time we bundle, we bundle by the same amount to make it easier to keep track of the grouping process. Therefore, the number of dots in each **bundle** (including the leftover individual dots) is 5^0, 5^1, or 5^2. If there were more dots, we could continue to group dots into bundles with 5^3, 5^4, 5^5, etc.

This method for keeping track of numbers with bundles gives rise to the decimal system. In the decimal system, we create bundles from powers of 10:

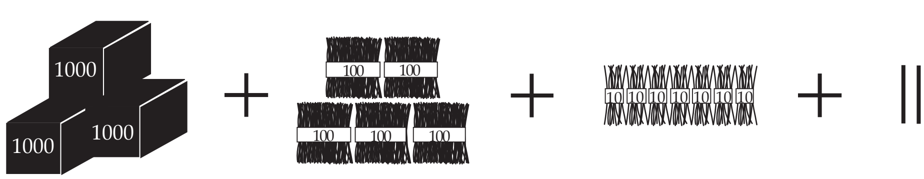


10 x 1 = 10   10 x 10 = 100   10 x 100 = 1000

Let's take a look at how these bundles give us a way to write numbers. Suppose you give your friend 3572 twigs to count. Your friend first makes 357 complete bundles of 10^1 twigs, with $2 = 2 \cdot 10^0$ twigs leftover. Of those 357 bundles of 10^1, your friend makes 35 complete bundles of 10^2 twigs each, leaving only 7 bundles of 10^1 left. Next, he groups together 3 whole bundles of 10^3 twigs, leaving 5 bundles of 10^2 left. With fewer than 10 bundles of 10^3, your friend stops bundling.

At this point, your friend has $3 \cdot 10^3$ twigs grouped in the largest bundles, another $5 \cdot 10^2$ twigs grouped in the next largest bundles, another $7 \cdot 10^1$ twigs bundled together, and $2 \cdot 10^0$ twigs leftover from the grouping process. He easily tallies the total number of twigs:

$$3 \cdot 10^3 + 5 \cdot 10^2 + 7 \cdot 10^1 + 2 \cdot 10^0 = 3000 + 500 + 70 + 2 = 3572.$$



3 x 1000 + 5 x 100 + 7 x 10 + 2 x 1 = 3572

Let's call quantities such as $3 \cdot 10^3$, $5 \cdot 10^2$, $7 \cdot 10^1$, and $2 \cdot 10^0$ **digit bundles** because we can use the coefficients of the powers of 10 (3, 5, 7, and 2) as digits with which to write the decimal **numeral** that represents the sum of the digit bundles.

Definition: A **numeral** is a symbol used to represent a number.

In the decimal number system, we use ten digits to build the numerals that represent numbers. The decimal number system is an example of a **base number system**. Because the decimal system groups quantities in powers of 10, we also refer to the decimal number system as **base 10**.

Definition: Systems for writing numbers using a select list of digits to represent digit bundles are known as **base number systems**.

When bundling in base 10, we use the 10 digits 0 through 9 inclusive. When we have 10 of a bundle, we bundle them into a larger bundle. Likewise, the base of any number system determines the number of digits that can be the coefficients of digit bundles. While $2 \cdot 3^2$ is a base-3 digit bundle, the quantity $5 \cdot 3^2$ is not because we can bundle 3 of the bundles of size 3^2 to create a bundle of size 3^3, so

$$5 \cdot 3^2 = 1 \cdot 3^3 + 2 \cdot 3^2,$$

where $1 \cdot 3^3$ and $2 \cdot 3^2$ are base 3 digit bundles. Here are some other examples of digit bundles:

$$\begin{array}{ccc} 1 \cdot 2^5 & 0 \cdot 9^2 & 6 \cdot 7^{16} \\ 3 \cdot 4^2 & 7 \cdot 9^7 & 5 \cdot 12^4 \end{array}$$

Exercises

8.2.1 Write each of the following sums of digit bundles as decimal numerals.

(a) $2 \cdot 10^2 + 5 \cdot 10^1 + 8 \cdot 10^0$

(b) $5 \cdot 10^4 + 0 \cdot 10^3 + 9 \cdot 10^2 + 1 \cdot 10^1 + 7 \cdot 10^0$

(c) $3 \cdot 8^2 + 7 \cdot 8^1 + 3 \cdot 8^0$

(d) $2 \cdot 4^3 + 3 \cdot 4^2 + 0 \cdot 4^1 + 2 \cdot 4^0$

Sidenote: While the ancient Egyptian number system grouped values of integers into bundles of powers of 10, it simply repeated symbols to represent numbers of bundles. Using digits to represent numbers of bundles allows us to write numbers more compactly. It also allows us to perform arithmetic more easily by lining up digits by their place values for addition, subtraction, and multiplication.

8.3 Base Numbers

Let's explore base number systems other than the ordinary base 10 that we're used to.

Problems

Problem 8.2: Fill in each blank with either 0, 1, 2, 3, 4, or 5 to make true statements.

$$\begin{array}{rcccccccc}
63 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 4\ } \cdot 6^1 & + & \underline{\ 3\ } \cdot 6^0 \\
64 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
65 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
66 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
67 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
68 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
69 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
70 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
71 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
72 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
73 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
74 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0
\end{array}$$

Problem 8.3: Suppose you make a list of the 64 smallest positive integers that can be written using only the digits 0 through 7 inclusive:

$$1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 20, \ldots$$

What would be the last number in your list?

Problem 8.2: Fill in each blank with either 0, 1, 2, 3, 4, or 5 to make true statements.

$$\begin{array}{rcccccccc}
63 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 4\ } \cdot 6^1 & + & \underline{\ 3\ } \cdot 6^0 \\
64 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
65 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
66 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
67 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
68 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
69 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
70 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
71 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
72 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
73 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0 \\
74 & = & \underline{\quad} \cdot 6^2 & + & \underline{\quad} \cdot 6^1 & + & \underline{\quad} \cdot 6^0
\end{array}$$

Solution for Problem 8.2: In this problem we rewrite base-10 numbers as sums of base-6 digit bundles.

We use the first line as a starting point. From there, we count up in the units digits and group bundles into larger bundles when necessary.

$$\begin{array}{rclclcl}
63 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 4\ } \cdot 6^1 & + & \underline{\ 3\ } \cdot 6^0 \\
64 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 4\ } \cdot 6^1 & + & \underline{\ 4\ } \cdot 6^0 \\
65 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 4\ } \cdot 6^1 & + & \underline{\ 5\ } \cdot 6^0 \\
66 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 5\ } \cdot 6^1 & + & \underline{\ 0\ } \cdot 6^0 \\
67 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 5\ } \cdot 6^1 & + & \underline{\ 1\ } \cdot 6^0 \\
68 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 5\ } \cdot 6^1 & + & \underline{\ 2\ } \cdot 6^0 \\
69 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 5\ } \cdot 6^1 & + & \underline{\ 3\ } \cdot 6^0 \\
70 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 5\ } \cdot 6^1 & + & \underline{\ 4\ } \cdot 6^0 \\
71 & = & \underline{\ 1\ } \cdot 6^2 & + & \underline{\ 5\ } \cdot 6^1 & + & \underline{\ 5\ } \cdot 6^0 \\
72 & = & \underline{\ 2\ } \cdot 6^2 & + & \underline{\ 0\ } \cdot 6^1 & + & \underline{\ 0\ } \cdot 6^0 \\
73 & = & \underline{\ 2\ } \cdot 6^2 & + & \underline{\ 0\ } \cdot 6^1 & + & \underline{\ 1\ } \cdot 6^0 \\
74 & = & \underline{\ 2\ } \cdot 6^2 & + & \underline{\ 0\ } \cdot 6^1 & + & \underline{\ 2\ } \cdot 6^0
\end{array}$$

□

In Problem 8.2, we write several numbers as sums of base-6 digit bundles. From these digit bundles we construct base-6 numerals. Just as they do in base 10, the coefficients of the digit bundles become the digits of the base-6 numerals:

$$\begin{array}{rclcrcl}
63 & = & 143_6 & \qquad & 69 & = & 153_6 \\
64 & = & 144_6 & & 70 & = & 154_6 \\
65 & = & 145_6 & & 71 & = & 155_6 \\
66 & = & 150_6 & & 72 & = & 200_6 \\
67 & = & 151_6 & & 73 & = & 201_6 \\
68 & = & 152_6 & & 74 & = & 202_6
\end{array}$$

Definition: When we write numerals using digits that represent the first b whole numbers $(0, 1, 2, \ldots, b-2, b-1)$, we say that we are writing numbers in **base** b. We call b the **base** of the number system. The base of a number system is also sometimes called the **radix** or **scale**.

In order to distinguish between numbers written in different bases, we write integers in number base systems other than 10 using the base as a subscript. For instance, we write the first 36 counting numbers in base 6 as

$$\begin{array}{rrrrrr}
1_6 & 2_6 & 3_6 & 4_6 & 5_6 & 10_6 \\
11_6 & 12_6 & 13_6 & 14_6 & 15_6 & 20_6 \\
21_6 & 22_6 & 23_6 & 24_6 & 25_6 & 30_6 \\
31_6 & 32_6 & 33_6 & 34_6 & 35_6 & 40_6 \\
41_6 & 42_6 & 43_6 & 44_6 & 45_6 & 50_6 \\
51_6 & 52_6 & 53_6 & 54_6 & 55_6 & 100_6
\end{array}$$

Since we typically use the decimal system to express numbers, we normally leave off the base-10 subscript. For instance, instead of writing 1354_{10}, we simply write 1354.

Concept: In the decimal system, we write numerals based on bundles of 10. In base 6, we write numerals based on bundles of 6. In base b, we write numerals based on bundles of b.

Let's take a look at some integers written in another number base. Here are the first 49 natural numbers written in base 7, where we use only the digits 0 to 6 inclusive:

$$\begin{array}{rrrrrrr} 1_7 & 2_7 & 3_7 & 4_7 & 5_7 & 6_7 & 10_7 \\ 11_7 & 12_7 & 13_7 & 14_7 & 15_7 & 16_7 & 20_7 \\ 21_7 & 22_7 & 23_7 & 24_7 & 25_7 & 26_7 & 30_7 \\ 31_7 & 32_7 & 33_7 & 34_7 & 35_7 & 36_7 & 40_7 \\ 41_7 & 42_7 & 43_7 & 44_7 & 45_7 & 46_7 & 50_7 \\ 51_7 & 52_7 & 53_7 & 54_7 & 55_7 & 56_7 & 60_7 \\ 61_7 & 62_7 & 63_7 & 64_7 & 65_7 & 66_7 & 100_7 \end{array}$$

Aside from the fact that the dimensions of the grid above are 7×7, we know that there are 49 integers in the list because it counts up to the integer

$$100_7 = 1 \cdot 7^2 + 0 \cdot 7^1 + 0 \cdot 7^0 = 49.$$

Note that we write negative numbers in bases other than 10 too. For instance, we count down from 0 in base 7:

$$0_7,\ -1_7,\ -2_7,\ -3_7,\ -4_7,\ -5_7,\ -6_7,\ -10_7,\ -11_7, \ldots$$

Problem 8.3: How do we write the decimal integer 64 in base 8?

Solution for Problem 8.3: Since $64 = 8^2 = 1 \cdot 8^2 + 0 \cdot 8^1 + 0 \cdot 8^0$, we get $64 = 100_8$. $\square$

Sidenote: Most computer information is stored in base 2, also known as **binary**. The fact that base-2 numbers have only two digits, 0 and 1, makes them easier to use in circumstances where someone would want to calculate by using a system of on-off switches (0 for "off", 1 for "on"). Computers are often made up of a series of binary switches.

Exercises

8.3.1 Write each of the following sums as a single base-8 numeral.

(a) $4 \cdot 8^1 + 3 \cdot 8^0$.

(b) $3 \cdot 8^2 + 1 \cdot 8^1 + 0 \cdot 8^0$.

(c) $7 \cdot 8^3 + 6 \cdot 8^2 + 0 \cdot 8^1 + 1 \cdot 8^0$.

(d) $5 \cdot 8^5 + 2 \cdot 8^2$.

8.3.2 Write each of the following sums as a single base-4 numeral.

(a) $3 \cdot 4^2 + 2 \cdot 4^0$.

(b) $1 \cdot 4^4 + 1 \cdot 4^3 + 2 \cdot 4^2 + 3 \cdot 4^1 + 1 \cdot 4^0$.

(c) $3 \cdot 4^5 + 1 \cdot 4^4 + 3 \cdot 4^3 + 3 \cdot 4^2 + 1 \cdot 4^1 + 3 \cdot 4^0$.

(d) $2 \cdot 4^7 + 1 \cdot 4^5 + 1 \cdot 4^3 + 3 \cdot 4^2 + 3 \cdot 4^0$.

8.3.3 Write the first 16 counting numbers in base 4.

8.3.4 Write the first 36 counting numbers in base 5.

8.3.5 Write the first 49 counting numbers in base 6.

8.3.6 If we begin writing counting numbers in base 3, how would we write each of the following numbers in our list?

(a) The eighth
(b) The ninth
(c) The tenth
(d) The twelfth
(e) The thirteenth
(f) The fourteenth
(g) The twenty-sixth
(h) The twenty-seventh
(i) The thirty-seventh

8.4 Base Number Digits

So far we've used no more than the 10 digits we use when writing numbers in decimal form. We use fewer of them for bases smaller than 10. But we can bundle integers in bases larger than 10 too!

Problems

Problem 8.4: If we want to write integers in base 12, we'll need 2 more digits in addition to the 10 we already use. How can we write integers in base 12?

Sidenote: Some base number systems are common enough that we give them their own names. Here are some of the most commonly used names for base number systems:

Base	Number System
2	binary
3	ternary
4	quaternary
5	quinary
6	senary
7	septenary
8	octal
9	nonary
10	decimal
11	undenary
12	duodecimal
16	hexadecimal
20	vigesimal
60	sexagesimal