Life of Fred®

Real Analysis

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Polka Dot Publishing

A Note to Classroom Students and Autodidacts

Real analysis is one of the central building blocks for later mathematics. Because of that, real analysis is a mandatory part of the education of every math major in virtually every university.

There are three central ideas in real analysis:

- ✯ the real numbers
- ✯ functions
- ✯ limits.

You will learn more about these three topics than you ever learned in algebra or calculus.

If this is your first *Life of Fred* book, you might be in for a little bit of a shock. Most real analysis textbooks look alike. They are all very businesslike: page after page of . . .

theorems,

definitions, and

lines of equations.

They "get the job done" because they "cover the material." The student memorizes what a Cauchy sequence is and can recite the Bolzano-Weierstrass theorem, but will forget them ten days after the final exam.

In contrast, *Life of Fred: Real Analysis* is fun. I, your author, can't help it. I refuse to write a stuffy book. (This is my 54th unstuffy book.) Who wants a life without some giggles and dancing?

With my best wishes,
Stan

A Note to Teachers

T*his book covers all the standard topics in real analysis*. There is an emphasis on creating proofs—not just memorizing theorems and definitions.

THIS BOOK HAS SEVERAL DRAWBACKS.

1. The solutions to all of the 113 exercises are supplied. The students can cheat like crazy on their homework. (Of course, they will die when they face your exams.) The advantage to my supplying all the answers is that you will not have to spend class time going over the homework problems.

2. This book is not dignified. How will you be able to face your fellow math professors if you adopt a textbook with such a silly title?

3. Your students will get fat, because they will have more money to spend on pizza after they shout for joy in the bookstore when they discover that you have adopted an inexpensive real analysis book.

4. Your students will be spoiled. They will expect that all upper-division math books are this delightful.

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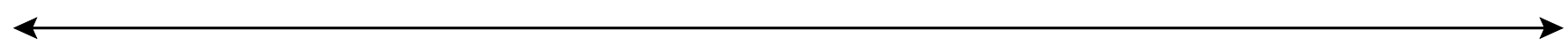
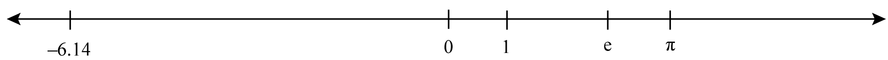
Chapter One
The Real Numbers

Fred is in love with the real numbers, ℝ. Just mention ℝ to him and he will tell you a thousand reasons why they are beautiful. First, he'll draw the real number line.



He will tell you how long and slender the real number line is. It goes on forever in both directions and is less than a millionth of an inch thick.*

Fred would name some of his favorite real numbers and mark them on the real number line.



When asked, not many people would say that –6.14 is one of their favorite numbers. Fred likes it because he was born 6.14 years ago.

He loves the real numbers because there are so many of them. There is an uncountably infinite number** of them. They are all fun to play with. Play with? Yes. If you are going to be a mathematician, enjoying playing with the real numbers is essential.

> *Occupational Notice*
>
> *If you are going to be a biologist, you have to enjoy cutting up dead things.*
>
> *If you are going to be a cook, you have to enjoy cutting up dead things and eating them.*
>
> *Mathematics is nicer.*

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*For those who prefer the metric system, we note that one inch is approximately 2.540 centimeters. So "a millionth of an inch" should be replaced by "0.000001/2.540 cm" which is approximately 0.0000003937007874015748031496062992126 cm.

***Uncountably infinite* is larger than *countably infinite*. We'll explain that in a minute. We are just getting started.

Fred got out his toy airplane. He was going to play with $\mathbb{R}$.

Hold it! Stop! I, your reader, have a zillion questions. [*] ***First of all, who is this Fred guy?***

Here's his background in case you have never read any of the other Fred books: Fred is 6.

I knew that. You told me that on the previous page.

Sorry. I see that you were paying close attention. Fred is a professor at KITTENS University. He had a rough childhood and left home at the age of six months. (This is chronicled in the first nine chapters of *Life of Fred: Calculus.*)

He lives with his doll Kingie in room 314 in the Math Building on the KITTENS campus. Now you know everything necessary for me to continue the story.

I said I had a zillion questions. My second question is more important. Are you going to define the real numbers as just something that is long and skinny. That's a rotten definition. A piece of spaghetti is long and skinny—and it's not the real numbers.

I'm going to ask for a little patience on your part. I was only one page into explaining why Fred loves $\mathbb{R}$, and you have spent a half page with interruptions. Do you remember the expression *festina lente*?

I never studied Italian.

It's not Italian. It's Latin. *Festina lente* means "make haste slowly." Often the best way to get something done is to allow a little untightening. We want to learn real analysis not just get through it.

Right! That's why I bought this book and am reading it. What's your point?

Plllllleeeeeaaaaasssseeee. This is not a horse race. I could pile all of real analysis into 50 pages—just listing theorem after theorem and definition after definition and proof after proof. You could sit down and read it in one day. And you would learn *nothing*. You are not a robot who learns as fast as its optical sensors can scan a page.

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[*] The *Life of Fred* books are the only books where you can talk back to the author.

I'll accept that. But could you tell me what the real numbers are before Fred gets out his toy airplane? Please.

Okay. Here are three different approaches to ℝ.

First Approach to ℝ:

This is the axiomatic approach. We start with five axioms (assumptions) about the natural numbers ℕ. We define addition in the natural numbers and prove a bunch of theorems about ℕ, such as 2 + 2 = 4. (We did this on page 58 of *Life of Fred: Five Days.*)

Then we define the integers ℤ. Then the rational numbers ℚ.

ℕ = {1, 2, 3, 4, 5, . . .}.

ℤ = { . . . –3, –2, –1, 0, 1, 2, 3, . . .}.

ℚ = { x | x = a/b where a, b ∊ ℤ and b ≠ 0}.

{ x | . . . } means "the set of all x such that . . .".

∊ means "is an element of." 72 ∊ ℕ

And the definition of ℝ (in *Life of Fred: Five Days*) is ultimately grounded in those original five axioms for ℕ.

You used the axiomatic approach when you studied high school geometry. You started with the geometry axioms (also called postulates) and proved a bunch of theorems. The first axiom was "Two points determine a unique line."

Second Approach to ℝ:

The second approach is real simple. The real numbers are any numbers that can be written as an unending decimal.

1 can be written as 1.0000000000000000000. . . .

$\frac{3}{5}$ can be written as 0.6000000000000000. . . .

$\frac{1}{3}$ can be written as 0.3333333333333333333. . . .

π can be written as

3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117067982 1
480865132823066470938446095505822317253594081284811174502841027019385211055596446229489549303819644288109
756659334461284756482337867831652712019091456485669234603486104543266482133936072602491412737245870066063
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520353018529689957736225994138912497217752834791315155748572424541506959508295331168617278558890750983817
546374649393192550604009277016711390098488240128583616035637076601047101819429555961989467678374494 48. . . .

$\sqrt{2}$ can be written as 1.4142135623730950488016887242097. . . .

Those decimals that repeat are rational numbers ($\mathbb{Q}$).
Those decimals that do not repeat are irrational numbers.

$\frac{1}{7}$ is rational. As a decimal it is 0.142857 142857 142857 142857 142857 142857 142857 142857 142857 142857 142857 142857 142857 142857 142857 142857. . . . (I added a little extra space so that you could see the repeat more easily.)

Hey! I like that. It's clean and neat. Now I know what $\mathbb{R}$ is.

Third Approach to $\mathbb{R}$:

We just assume all the facts about the real numbers that you learned in algebra.

Um. I can't find my algebra book. Would you care to list "all those facts"?

Yikes! There's a million of them. We'll be here all day if I write them all out.

List them. I bought this book. You do some work.

Okay. But I'm going to use a smaller font so that the list doesn't take up half of this book. You know all of this stuff anyway.

1. Commutative laws: a + b = b + a and ab = ba
2. Associative laws: (a + b) + c = a + (b + c) and (ab)c = a(bc) (Are you bored yet?)
3. Distributive law: a(b + c) = ab + ac
4. For any real number r, r + 0 = r and 1r = r
5. For any real number r, there is a real number s such that r + s = 0.

   If we start with 38.9, then there exists –38.9 such that 38.9 + (–38.9) = 0.
6. For any nonzero number r, there is a real number s such that rs = 1.

   If we start with 7.4, there is a real number 1/7.4 such that (7.4)(1/7.4) = 1.

   If we start with π, there is a real number 1/π such that π(1/π) = 1.

7 The Triangle Inequality: $|a + b| \leq |a| + |b|$ ("The most important inequality in math.")

8 The Triangle Inequality for Subtraction: $\big||a| - |b|\big| \leq |a - b|$

9 Transitive properties: If $a = b$ and $b = c$, then $a = c$. If $a < b$ and $b < c$, then $a < c$.

10 If $a < b$, then $a + c < b + c$.

11 If $a < b$ and $c > 0$, then $ac < bc$.

12 etc.

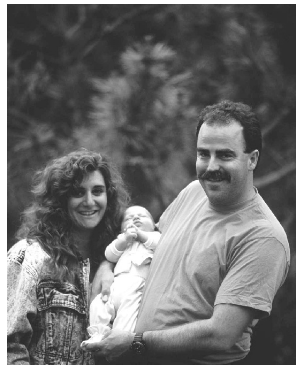
Intermission

Mathematics after calculus is a lot different. In arithmetic, algebra, trig, and calculus, you were finding answers. You did a bunch of computing and figured out that it took her 15 hours to paint the house when working alone or that the area was 27 square feet. Engineers liked that kind of stuff.

In upper division math we spend our time understanding concepts and proving that certain things are true.

Instead of having 30 exercises to grind through, there are puzzles to solve. Often, they are in the form of, "Can you show that this is true?" Creativity—not computation—becomes the theme song.

MEET MAX



Maxine and Maxwell
(Minimum is the in the middle)

The **maximum** member of a set is the largest member of that set.

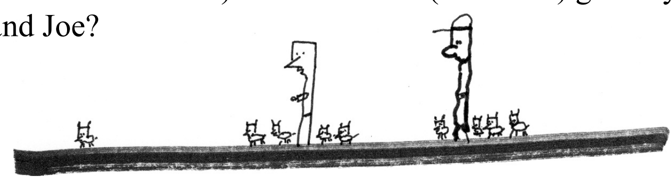
Puzzle #30: Complete this definition using lots of math symbols. M is the maximum element of set B iff. . . .

Puzzle #31: Does every infinite set of real numbers have a maximum?

Puzzle #32: Does every finite set of real numbers have a maximum?

Puzzle #33: Give an example of a finite set that does not have a maximum.

We'll need max to answer this question: Can a sequence have two limits? (That's the mathematical equivalent of the old question: Can a man have two masters?) Can the cows (the terms) get very close to both Fred and Joe?



Theorem: If $\lim a_n = F$ and $\lim a_n = J$, then $F = J$.
In English: Limits are unique.

Informal proof: Fred and Joe have to be standing some distance apart. Say they're 4Å apart. (That's four angstroms.*) Suppose we put fenceposts around Fred and around Joe that are 2Å from each of them.

Since we are given that the limit of the sequence is Fred, we know that only a finite number of cows are outside of Fred's little enclosure. But there are an infinite number of cows inside Joe's enclosure. Contradiction. ⊠

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*Everybody wants me to include lots of metric system units in my books since roughly 99% of all countries in the world are on the metric system nowadays.

Å = one ten-billionth of a meter = 10^{-10} m.

Formal proof: Assume $F \neq J$. Then $|F - J| > 0$. Let $\varepsilon = |F - J|$.

$(\exists N \in \mathbb{N})(\forall n > N)\ |a_n - F| < \varepsilon/2$	definition of $\lim a_n = F$
$(\exists M \in \mathbb{N})(\forall n > M)\ |a_n - J| < \varepsilon/2$	definition of $\lim a_n = J$
Let $\mathcal{M} = \max\ \{M, N\}$.	Here's where we use max. We know that the maximum exists because $\{M, N\}$ is a finite set consisting of just two members.

$(\forall n > \mathcal{M})\ |a_n - F| < \varepsilon/2$ and $|a_n - J| < \varepsilon/2$

$|a_n - F| + |a_n - J| < \varepsilon$	algebra
$|F - a_n| + |a_n - J| < \varepsilon$	property of absolute values, namely, $|x - y| = |y - x|$
$|F - a_n + a_n - J| < \varepsilon$	$|F - a_n + a_n - J| \leq |F - a_n| + |a_n - J|$ by what we called "The most important inequality in math." On pages 58, 68, 70, 134, 136, 213, 269, we use . . . The Triangle Inequality: $|a + b| \leq |a| + |b|$
$|F - J| < \varepsilon$	algebra. This line contradicts the first line of the proof. ⊠

In our preview of the coming attractions for this chapter, we had four topics: **bounded sequences, convergent sequences, subsequences,** and **Cauchy sequences.** It's time to tie the first two topics together.

Theorem: If $\{a_n\}$ is convergent, then it is bounded.

One student of mine once wrote in his notes: *Converg ⇒ bounded.*

Proof.

1.	$\{a_n\}$ is convergent	given
2.	$\lim a_n = L$	def of convergent
3.	$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})\ |a_n - L| < \varepsilon$ for every $n > N$	def of $\lim a_n$
4.	$(\exists N \in \mathbb{N})\ |a_n - L| < 1$ for every $n > N$	letting $\varepsilon = 1$

5. Let $B = \max \{|a_1|, |a_2|, |a_3|, \ldots, |a_n|, 1\}$ — This is a finite set and finite sets always have a maximum element by puzzle #32

6. $(\forall n \in \mathbb{N})\ |a_n| < B$ — lines 4 and 5 ⊠

Puzzle #34: If $\{a_n\}$ is bounded, must $\{a_n\}$ be convergent?
Puzzle #35: If $\{a_n\}$ is bounded, must $\{a_n/n\}$ be convergent?
Puzzle #36: Show that if $\{a_n\}$ and $\{t_n\}$ are convergent and if $(\forall n)\ a_n < t_n$, that does *not* imply that $\lim a_n < \lim t_n$.

❀ ❀ ❀

In the footnote on the first page of this chapter, I mentioned that there is just one more property of $\mathbb{R}$ that we will need. You already have

the commutative laws ➺ $a + b = b + a$ and $ab = ba$
the associative laws ➺ $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$
the distributive law ➺ $a(b + c) = ab + ac$
the identity laws ➺ $(\forall r \in \mathbb{R})\ r + 0 = r$ and $1r = r$
the inverse laws ➺ $(\forall r \in \mathbb{R})(\exists s \in \mathbb{R})\ r + s = 0$ and
$(\forall r \in \mathbb{R})(r \neq 0)(\exists s \in \mathbb{R})\ rs = 1$

(Time out! The above laws establish that the real numbers are a **field**. Fields are discussed in detail in Modern Algebra, which is another upper division math course.)

Both $\mathbb{R}$ and $\mathbb{Q}$ (the rational numbers) are fields.

the Triangle Inequalities ➺ $|a + b| \leq |a| + |b|$ and $||a| - |b|| \leq |a - b|$
the **Law of Trichotomy** ➺ exactly one of these is true:
$a < b$, $a = b$, or $a > b$.

What makes $\mathbb{R}$ so special is the AoC. $\mathbb{Q}$ doesn't have the AoC.
Okay. I, your reader, want to know. What's the AoC?
I have to sneak up on the AoC. Pussyfoot up to it—as you call it.

Definition: Set $A \subset \mathbb{R}$ has an **upper bound** B iff $a \in A \Rightarrow a \leq B$.
I could have guessed that! Now what's this AoC?

Patience, please. If you want to hurry, just read faster.

Definition: $\mathcal{B}$ is the **least upper bound** of a set $A \subset \mathbb{R}$ iff $\mathcal{B}$ is an upper bound for A and for every other upper bound B, $\mathcal{B} < B$.

Puzzle #37: Prove that a set A cannot have two different least upper bounds.

Now I can present the AoC.
Okay. Spit it out.

The AoC: Every subset of the real numbers that has an upper bound in $\mathbb{R}$, also has a least upper bound in $\mathbb{R}$.

A little more spit please.

AoC stands for the **Axiom of Completeness**. It says that the real numbers are complete.

The rational numbers do not have an AoC. For example, the set of rational numbers {3, 3.1, 3.14, 3.141, 3.1415, 3.14159, . . . } has an upper bound of 7, but there is no rational number which is the least upper bound for this set. (In $\mathbb{R}$, the least upper bound would be π.)

The AoC means that there are no holes in the real number line. When we were in the rational numbers tip-toeing 3, 3.1, 3.14, 3.141, 3.1415, 3.14159, . . . there was a big hole at π.

❁ ❁ ❁

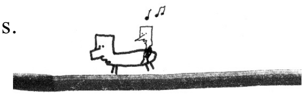
Now that you have the Axiom of Completeness for the real numbers, you are ready for the last theorem of the **convergent series** section of this chapter . . .

The Rabbit and the Wall theorem

You gotta be kidding.
No I'm not. It's based on a true story.

The Story / *The Math*

Fred was out on the Kansas plain with his horse he called Pferd. (Pferd is a German word.) He was playing his ukelele and singing cowboy songs, just as he had seen in the movies.



too much sugar

Fred spotted this rabbit who was nuts.

It would hop up and down in the same spot for hours.

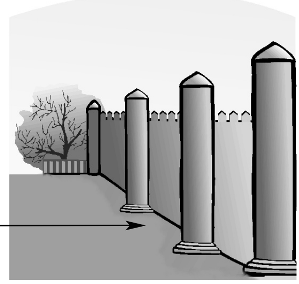
$a_1, a_2, a_3 \ldots$ the hops

Sometimes it would hop forward a little—always toward the east.

$(\forall n)\ a_n \le a_{n+1}$
a non-decreasing sequence

Fred thought to himself *If this rabbit wants to hop all the way to the Atlantic ocean, that's fine.*

Then he noticed that there was a giant wall right in the path of the rabbit. It would never be able to pass it.



path of rabbit

a bounded sequence
$(\exists B \in \mathbb{R})(\forall n)\ a_n \le B$

Fred knew what was going to happen, because he knew the Rabbit and the Wall theorem.

Theorem: Every bounded, non-decreasing sequence is convergent.

This did not mean that the rabbit would ever hit the wall. What it means is that at some point that rabbit will basically be hopping up and down at some place.

Note to reader: **Every sentence on this page is absolutely true with only one exception.** (← That's the exception).

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