

Life of Fred[®]

Five Days of Upper Division Math:

Set Theory

Modern Algebra

Abstract Arithmetic

Topology

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Polka Dot Publishing

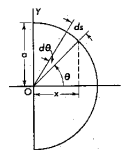
A Note to Readers

I am a mathematician. That wasn't always the case. In high school the four years of math were easy compared with reading *David Copperfield* (didn't like) and *Moby Dick* (liked), typing papers (10–20 pages), or memorizing history facts. But that didn't make me a mathematician.

The first two years of college calculus were not pleasant. The teachers made us memorize stuff and gave us no real reason why we might ever use it. Opening my old calculus textbook* at random I read:

146. Centroid and Moment of Inertia of Arc

Let the arc AB of a curve be divided into n parts as show in Figure 180, and let (x_k, y_k) be any point on the k th segment of arc $\bar{A}s_k$. In accordance with the definition of centroids for areas and volumes, we define the centroid of an arc as the point (\bar{x}, \bar{y}) determined by the relations $\bar{s}\bar{x} = \lim \sum x_k \bar{A}s_k$ [etc.]



Taught that way, calculus was definitely not fun. My four semester grades were A, C, B, and C.**

Needless to say, I wasn't a mathematician after those two years of calculus.

At that time, there were three reasons I chose mathematics as my major: ① The grading wasn't subjective. If you got the right answer, the teacher couldn't argue. A friend of mine was a political science major. He supported the individual over the state and 99% of the faculty were statisticians. On one oral Ph.D. exam he received an A and two F's. He had to change universities in order to get his doctorate. ② Being a math major offered much better employment opportunities than any major with the word *studies* in it. ③ Math majors don't have to write long term papers.***

* p. 331 of Thurman S. Peterson's *Calculus with Analytic Geometry*

** Years later, when I taught college calculus, all the tests I gave were open book (no more memorizing!), and my lectures included lots of "Fred" to illustrate how calculus is relevant in everyday life. Each year I taught, I included more Fred.

*** Life is "slightly" unpredictable. This is my 35th book. Five of them have been more than 540 pages long. *Life of Fred: Trig Expanded Edition* was only 496 pages. Writing about Fred is a pure joy.

Of all my relatives, I was the first one to get a college education.
No one could offer me the good news:

*Once you get to
upper division pure math,
the world changes
for the better.*

All the engineering majors who sat next to you in calculus have gone off to Engineering Land. They are off computing centroids of arcs and building bridges, chronometers, and skyscrapers.

For the happy few who enter the magic worlds

🍒 of set theory

🍒 of modern algebra

🍒 of abstract arithmetic

🍒 of topology

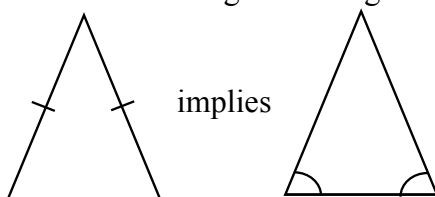
there are

- ◆ no “word problems”
- ◆ no papers to write
- ◆ no concrete applications
- ◆ no dates or formulas to memorize.

How many ounces of Captain Mousebait cereal (7% flour) should you mix with Sergeant Sugar cereal (11% flour) to obtain 300 ounces of cereal with 8% flour? This is an actual problem, which can be found on page 28 in *Zillions of Practice Problems for Beginning Algebra*. (answer = 225 ounces.)

Instead, there are simply puzzles to solve. Often the puzzles ask the student to prove things. And just like in geometry, there can be more than one way to create a proof.

In *Life of Fred: Geometry*, we showed four different proofs that the base angles of an isosceles triangle are congruent.



One way was to draw a segment from the top vertex to the midpoint of the base (a median) and show the triangles are congruent by SSS. A second way was to draw the angle bisector from the top angle and show the triangles are congruent by SAS.

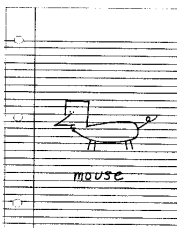
Instead of computing answers like the folks over in Engineering Land, you will be engaged in pure thought. Both activities can be hard work, but they are *different* kinds of work.

I became a mathematician during my junior and senior years at the university. I can't point to a particular instant in time when this happened, but I remember the joy of taking five math courses ☺☺☺☺☺ in my last undergraduate semester.

I have never seen any other book attempt to do what we are going to do here: The first five days in four upper division math courses taught by our master teacher Fred Gauss. Some of the puzzles (proofs) will be easy and some will be **hard**. If they were all easy, it really wouldn't be as much fun.

There will be no final exam, no grades, and no competition with other students.

I have been looking forward to writing this book for more than a decade. I love set theory, modern algebra, abstract arithmetic, and topology. One thing that has held me back is that there will be no bank robberies, no animals, no C.C. Coalback, and no boxing matches in this book. These things were easy to include when I wrote all the books from *Life of Fred: Apples* up through *Life of Fred: Calculus*. The former things are passed away. I am making all things new.*



What we'll miss

THREE PREREQUISITES FOR THIS BOOK

1. You gotta know what *prerequisite* means.
2. Two decent courses in high school algebra that included such things as unions of sets, math induction, the associative property, one-to-one functions, inverse functions, and multiplying matrices. One decent course in geometry that included lots of proofs.
3. The math in this book is the first parts of upper division mathematics for math majors. In a university setting the students in these classes are battle hardened with two years of calculus. They are used to having to work hard to understand the new material. They do not fold up into a little ball and blow away when they don't instantly understand a new concept. The third prerequisite is that you are not a fluff ball.

With my best wishes,


*I stole those two sentences from another book.

Contents

MONDAY

Set Theory 15

cardinality of a set 16, set builder notation 16, union and intersection 17, subset 17, naive set theory 17, modus ponens 18, seven possible reasons to give in a math proof 18, the high school geometry postulates are inconsistent 19, every triangle is isosceles 19, normal sets 22

Modern Algebra 23

math theories 23, definition of a theorem 24, six properties of equality 25, binary operations 25, formal definition of a binary operation 26, formal definition of a function 26, definition of a group 27, \forall and \exists 27, right cancellation law 27, left inverses 27, commutative law 28

Abstract Arithmetic 29

circular definitions 30, unary operations 31, successor function **S** 31, natural numbers 31, the five Peano postulates 32, mathematical induction 33

Topology 36

topology is all about friendship 36, listing all possible subsets 37, open sets 37, the discrete topology 37, the three axioms of a topology 38, models for a topology 40, open intervals 40

TUESDAY

Set Theory 43

axiom of extensionality 46, propositional functions 47, Zermelo-Fraenkel axiom #2 (axiom schema of specification) 48

Modern Algebra 49

three examples of non-commutative groups 51, uniqueness of right inverses and right identities 55

Abstract Arithmetic 56

no number can equal its successor 56, definition of $+$ in \mathbb{N} 57, recursive definitions 57, proving $2 + 2 = 4$ 58

Topology 61

the rationals are dense in the real numbers 62, topology of X when X is small 65, limit points 65, standard topology for \mathbb{R} 66, closed intervals 66

WEDNESDAY

Set Theory 67

ZF #3, the axiom of pairing 68, ZF #4, the axiom of union 69

Modern Algebra 71

$(a^{-1})^{-1} = a$ 71, If a and b are members of a group and if $a^2 = e$ and if $b^2a = ab^3$, then $b^5 = e$. 73, defining cardinality in terms of 1-1 onto functions 75, group isomorphisms 75

Abstract Arithmetic 79

recursive definition of multiplication in \mathbb{N} 80, proof of the distributive law 80, definition of n^m in \mathbb{N} 81, definition of the least member of a set in \mathbb{N} 83, strong induction 83, total binary relations 84

Topology 85

derived sets 85, closed sets 85, set subtraction 85, closed set axioms for a topology 86, closure of a set 87, index sets 87

THURSDAY

Set Theory 89

ZF #5, the power set axiom 89, Cartesian products, relations, and functions 93, domains, codomains, and ranges 93, one-to-one onto functions and the cardinality of sets 94

Modern Algebra 95

groups of low order 95, Klein four-group 96, If a and b are members of a group and if $ba^2 = ab^3$ and if $a^2b = ba^3$, then $a = e$. 98, subgroups 99

Abstract Arithmetic 101

partition of a set 101, equivalence relations 102, equivalence classes 103, defining the integers as equivalence classes 105, integer addition 106, integer multiplication 107, well-defined 108, $<$ in the integers 109, integer subtraction 109, proof that a negative times a negative gives a positive answer 109

Topology 110

limit point definition of continuous functions 111, continuous functions and open sets 113

FRIDAY

Set Theory 114

ZF #6, axiom of replacement 114, ZF #7, axiom of infinity 116, inserting all of abstract arithmetic into set theory 117, ZF #8, axiom of foundation 118, Schröder-Bernstein theorem 119, inaccessible cardinals and other big cardinals 121, metamathematics 121

Modern Algebra 123

cosets 124, cosets are either equal or disjoint 125, Lagrange's theorem 125, groups, semigroups, monoids, abelian groups, rings, fields, and vector spaces 126

Abstract Arithmetic 128

the rational numbers defined as equivalence classes 129, $+$, \times , $-$, and \div in \mathbb{Q} 130, ways not to define the real numbers 131, cuts in \mathbb{Q} 132, real numbers defined 132, most irrational numbers do not have nice names 134, the complex numbers 135

Topology 136

separated 136, connected 136, continuous image of a connected set is connected 137, open coverings 137, compact 137, continuous image of a compact set is compact 138, T_1 -, T_2 -, regular, T_3 -, normal, and T_4 -spaces 139

Solutions 140

Index 206

MONDAY

Set Theory

Prologue

It was the start of the summer classes at KITTENS University. The university president had told Fred that as long as he taught the required math courses, he was free to augment his schedule with any other classes he wished.

Fred was overjoyed. He had never been given such freedom. This was a chance to teach some junior- and senior-level math courses.

The president's secretary emailed the list of the courses he was required to teach:

8–9 Arithmetic
9–10 Beginning Algebra
10–11 Advanced Algebra
11–noon Geometry
noon–1 Trigonometry
1–2 Calculus
2–3 Statistics
3–3:05 Break
3:05–4 Linear Algebra

Fred was delighted. This was a lighter load than he had had in the spring semester. He added his four favorite upper division courses:

4–5 *Set Theory*
5–6 *Modern Algebra*
6–7 *Abstract Arithmetic*
7–8 *Topology*
8–9 Arithmetic
9–10 Beginning Algebra
10–11 Advanced Algebra
11–noon Geometry
noon–1 Trigonometry
1–2 Calculus
2–3 Statistics
3–3:05 Break
3:05–4 Linear Algebra

For a six-year-old experienced university professor like Fred, this would be a pleasant twelve-hour teaching schedule. The major difference for Fred would be that he would go jogging at 3 a.m. instead of at dawn as he had done for years.

4 a.m.

There were 300 students in the Archimedes auditorium classroom awaiting their master teacher. Two of Fred's best students, Betty and Alexander, were there.

The news had spread through the mathematics communities around the world that Fred was going to teach four upper division math courses for

MONDAY Set Theory

the first time. Many people instantly changed their summer plans and headed to KITTENS University. They filled the rest of the seats in the auditorium. The same people would be attending all four classes.

Fred entered. He had on his customary bow tie that he liked to wear when he was teaching, but had forgotten to change out of his jogging shorts. No one noticed. He waved hello and the room became silent.



Good morning. (← Fred's speech is in this font.)

This was Fred's time of day. At about 6 p.m. each evening he would be heading to bed to get his needed nine hours of sleep that every six-year-old needs. One of the students had a thermos with a liter of strong hot coffee. After he had drunk a little, he had a quart.

Some mathematicians have claimed that virtually every part of math could be ultimately based on set theory. It's a good place to start our day.

You have had high school math so you already know that a set is just any collection of things. The set containing \clubsuit and the number 8 can be written as $\{\clubsuit, 8\}$. Those curly parentheses are called **braces**. This is a left brace.

Fred wrote $\{$ on the blackboard.

$\{\clubsuit, 8\}$ and $\{8, \clubsuit\}$ are the same set. The order in which you list the elements of the set doesn't matter.

Please don't list the same member* of a set more than once. Don't write $\{\clubsuit, 8, \clubsuit\}$. It makes it hard to count the number of elements in a set if there are duplicates in the listing.

The **cardinality** of a set is the number of members in the set. The cardinality of the **empty set**, $\{\}$, is zero. The empty set is sometimes called the **null set** and is sometimes represented by the symbol \emptyset .

A second way to list a set is to use **set-builder notation**. If I wanted to list all the prime numbers that are less than a thousand, I could write $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733,$

* member of a set = element of a set

MONDAY Set Theory

739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997} or, using set-builder notation, write $\{x \mid x \text{ is a prime number less than } 1000\}$. This is read as, “The set of all x such that x is a prime number less than 1000.”

If I were to write $\{y \mid y \text{ is a prime number less than } 1000\}$ that would be the same set.

We abbreviate “is a member of” by \in . $8 \in \{\clubsuit, 8\}$

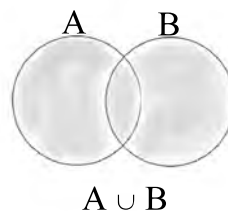
We abbreviate “is not a member of” by \notin . $9 \notin \{\clubsuit, 8\}$

If we have two sets, A and B , we define the **union** of A and B , $A \cup B$, as $\{x \mid x \in A \text{ or } x \in B\}$.

The **intersection** of A and B , $A \cap B$, is defined as $\{x \mid x \in A \text{ and } x \in B\}$.

Or in mathematics is the non-exclusive *or*. It means one or the other or both. Lawyers who want to indicate the non-exclusive *or* write *and/or*. Police who shout, “Stop or I’ll shoot” are hopefully using the exclusive *or*. You don’t want to stop *and* get shot.

The other thing we did in high school math was to draw Venn diagrams. We colored in circles.



We defined **subset**:

C is a subset of D , written $C \subset D$, if every element of C was in D .

In thirteen years of school—kindergarten through 12th grade—this was set theory. In mathematics, this material is called **naïve set theory**.*

The only thing wrong with naïve set theory is that it contained *contradictions*. You can prove a statement is true, and you can prove its opposite is also true. Once that happens, the game is over. Everything falls apart.

In logic, if you know that statement P is true and you also know that not- P is true, then *you can prove that anything is true*. In symbols, $(P \ \& \ \neg P) \Rightarrow Q$, where Q is any statement.**

* *Naïve* is pronounced nigh-EVE. Naïve = simple, unsophisticated. Coloring in circles is not really heavy-duty math.


** $\&$ = *and* \neg = *not* \Rightarrow = *implies*

MONDAY Set Theory

There are two famous theorems in logic. The first one is $(P \& \neg P) \Rightarrow Q$. ~~There shall not be any contradictions.~~

The second one is called **modus ponens**. If you know that statement P implies statement Q and you know that statement P is true, then you can infer that statement Q is also true. $(P \Rightarrow Q \& P) \Rightarrow Q$.

If your mother says, "If you do that, I'll ground you," and if you do that, then you know you'll be grounded.

Creating proofs is the heart of upper division math. One big reason you spent a year studying high school geometry was to learn how to prove things. It wasn't to learn area formulas or that the base angles of an isosceles triangle are equal. In the eighth grade you already knew that the opposite sides of a parallelogram  are equal. What you learned in geometry was how to *prove* that.

The rules for doing a proof are easy: ① Every line must have a reason that justifies that line, and ② the last line must be what you want to prove. The rules for most board games are much more complicated.

Here are some of the reasons we used in geometry. We use the same ones in upper division math.

1. Given
2. Postulate or axiom. (These words mean the same thing.)
3. Definition
4. Previously proven theorem
5. Beginning of an indirect proof
6. Contradiction in steps ____ and ____, and therefore the assumption in step ____ is false.
7. Cases

Reasons 5 and 6 are always paired together. In the beginning of an indirect proof, you assume the opposite of what you want to prove. Then you derive a contradiction. That contradiction indicates that your initial assumption was false.

How to Prove You Are Alive

- | | |
|-----------------------------------|--|
| 1. Assume I'm dead. | 1. Beginning of an indirect proof |
| 2. I couldn't be speaking to you. | 2. Definition of dead |
| 3. I am talking to you. | 3. |
| 4. I am alive. | 4. Contradiction in steps 2 and 3 and therefore the assumption in step 1 is false. |

MONDAY Set Theory

If the beginning assumptions (postulates or axioms) are true, then what you prove must also be true. Mathematics is a truth-generating machine.

Of course, if the postulates are inconsistent, then the whole system crashes, and you can prove anything.

Fred giggled a little at this point.

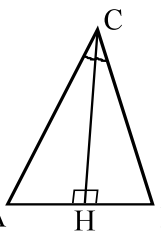
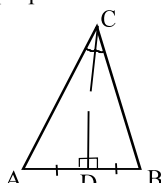
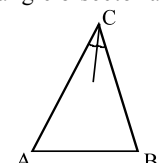
And what most geometry teachers and most geometry books fail to mention is that *The high school geometry postulates are inconsistent.*

A stunned silence fell over the audience. Everyone stopped writing and looked at Fred. Those who had not read *Life of Fred: Geometry* had no idea this was true. One of the students, Thomas, raised his hand and said, “I can’t believe that. Everyone knows that high school geometry is true. Unless you show me a contradiction—one I can see and understand—”

No problem. What if I prove that every triangle is isosceles?

Thomas laughed to himself. And Fred began.

Statement	Reason
1. Any old triangle ABC.	1. Given
2. Draw the angle bisector at C	2. By the angle measurement postulate $\angle C$ has a measurement between 0 and 180. And by the angle measurement postulate, there is an angle equal to half of that measurement. (Or, more simply, angle bisectors exist.)
3. Erect the perpendicular bisector of \overline{AB} .	3. Theorem: Every segment has a midpoint, and Theorem: You can erect a \perp to a line at any point on that line. (Or, more simply, every segment has a \perp bisector.)
4. The angle bisector and the \perp bisector are parallel.	4. Case 1 (One of two possibilities.)
5. The angle bisector is \perp to \overline{AB} .	5. Theorem: If a line (in this case \overline{AB}) is \perp to one of two parallel lines, it is \perp to the other.



MONDAY Set Theory

6. $\angle AHC \cong \angle BHC$

7. $\overline{CH} \cong \overline{CH}$

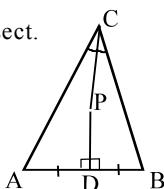
8. $\triangle AHC \cong \triangle BHC$

9. $\angle A \cong \angle B$

10. $\triangle ABC$ is isosceles.

11. The angle bisector and the \perp bisector are not parallel.

12. They intersect.



13. From the point of intersection, P, drop \perp s to \overline{AC} and \overline{BC} .

14. $PE = PF$

15. $AP = BP$

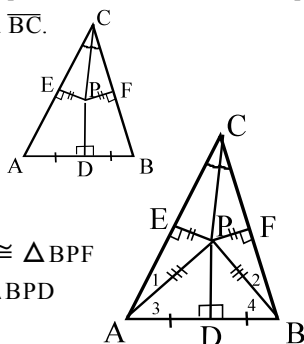
16. $\triangle APE \cong \triangle BPF$

$\triangle APD \cong \triangle BPD$

17. $\angle 1 \cong \angle 2$ $\angle 3 \cong \angle 4$

18. $\angle CAB \cong \angle CBA$

19. $\triangle ABC$ is isosceles.



6. Theorem: \perp lines form congruent right angles.

7. Theorem: Every segment is \cong to itself.

8. ASA

9. Definition of $\cong \triangle$.

10. Converse of the Isosceles Triangle theorem. (If the base \angle s are \cong , then the \triangle is isosceles.)

11. Case 2 (The only other possibility.)

12. Definition of not parallel.

13. Theorem: From any point you can drop a perpendicular to a line.

14. Theorem: Any point on an \angle bisector is equidistant to the sides of the angle.

15. Any point on a \perp bisector is equidistant from the endpoints of the segment.

16. Hypotenuse-leg theorem. (In any pair of right \triangle , if the hypotenuses and one pair of legs are \cong , then the \triangle are \cong .)

17. Definition of $\cong \triangle$.

18. Angle Addition postulate

19. Converse of the Isosceles Triangle theorem. (If the base \angle s are \cong , then the \triangle is isosceles.) \square

This marks the end of a proof. It's the same as Q.E.D.

There is nothing wrong with this proof *if* you accept the postulates of high school geometry. The problem is that high school geometry allows this to happen. The postulates allow this contradiction (and many others) to happen.

Thomas's life changed at this point. He had seen the broader vistas of upper division math.

MONDAY Set Theory

And naive set theory contains . . .

The entire classroom said, “No. No. No. It can’t be. Impossible. Incredible. No way.”

. . . contradictions.

The audience looked like they had been doused with a bucket of cold water.

Even with the little bit of set theory that I’ve described this morning, there’s enough to find a contradiction. And once you have a contradiction, $P \ \& \ \neg P$, you can prove anything. $(P \ \& \ \neg P) \Rightarrow Q$

Betty turned to Alexander and said the famous words from the “Wizard of Oz” movie: “I don’t think we’re in Kansas anymore.” Everyone in the audience fastened their seatbelts.* Every eye was on Fred.



First of all, let’s consider all those sets that are members of themselves.** Let A be the name of this set.

Then $A = \{x \mid x \in x\}$.

Is $A \in A$?


Obviously yes. It satisfies the definition: $\{x \mid x \in x\}$.

Most sets do *not* contain themselves as members. For example, the set of natural numbers, $\{1, 2, 3, 4, 5, \dots\}$. Or the set of all ducks.

Or the set of all fish that know the words to the fourth verse of our national anthem. Another name for this last set is the empty set, $\{ \}$, or \emptyset .

Oh! thus be it ever, when freemen shall stand
Between their loved homes and the war's desolation!
Blest with victory and peace, may the
heaven-rescued land
Praise the Power that hath made and preserved us a
nation.
Then conquer we must, when our cause it is just,
And this be our motto: "In God is our trust."
And the star-spangled banner in triumph shall wave
O'er the land of the free and the home of the brave!

* All of the classrooms that Fred teaches in have seatbelts. Boring teachers should have classrooms with pillows.

** Sets that contain themselves as members are fairly rare. Most sets do not contain themselves as a member. The set of your hands contains . It doesn’t contain any sets inside the braces.

One set that is a member of itself is the set of all sets mentioned in this book.

MONDAY Set Theory

Sets that do not contain themselves as members are called **normal sets**. Let's let B equal the set of all normal sets. $B = \{x \mid x \notin x\}$

Our hour is almost up. For Tuesday please create proofs for these two theorems.*

Theorem 1: If $B \in B$, then $B \notin B$.

Theorem 2: If $B \notin B$, then $B \in B$.

After you have proven both of these, you have established that

$B \in B$ iff $B \notin B$.

(iff = if and only if)

This is a contradiction.

Note to readers:

Answers to all of Fred's assignments are given in the back of this book.

Please do not just read the question and just turn to the answer. You won't learn very much if you do that.

Write out your answers first.

The fun part of this upper division math is solving the puzzles—not just learning “stuff.”

On Tuesday we will go beyond naive set theory.

I will present a list of set theory axioms that do not contain any contradictions.

I'll see you tomorrow.

* A theorem is a statement that has been or can be proven.

Index

- $2 + 2 = 4$ —the proof. 58
- abelian groups. 126
- accumulation point. 66
- addition in the natural numbers—
 definition. 57
- addition in the rational numbers
 129
- addition of integers. 106
- associative law in the natural
 numbers—the proof. . . 58
- axiom of extensionality. 46
- axiom schema of restricted
 comprehension. 48
- axiom schema of separation. . . 48
- axiom schema of specification
 48
- axioms of topology. 38
- binary operation. 25
- binary relation. 84
- total. 84
- braces. 16
- brother and sister theorems for
 continuous functions
 113
- cardinality of a set. . . 16, 94, 114
- Cartesian product. 93
- circular definitions. 30, 46
- closed sets. 85
- closed sets axioms for a topology
 86
- closure of a set. 87
- cluster point. 66
- codomain. 93
- commutative law for $+$ in the
 natural numbers—
 the proof. 59
- compact. 137, 138
- complement of set. 86
- complex numbers. 135
- connect the midpoints of sides of
 any quadrilateral. 49
- connected. 136, 137
- continuous functions. . . 110-113
- contrapositive statements. . . 158
- converse. 159
- coset. 124, 125
- Crimean War. 39
- cross multiplying. 200
- cut. 132-134
- cyclic group. 184
- De Morgan's laws. 86, 175
- dense. 62
- derived point. 66
- derived set. 85
- discrete topology. 37
- distributive property in the
 natural numbers. 80
- division in the rational numbers
 130
- domain. 93
- empty set. 16
- equality—six properties. 25
- equivalence class. 103
- equivalence relation. 102
- Euclid. 32
- every rational number is a
 repeating decimal. 62
- every triangle is isosceles—proof
 19, 20
- exponents in the natural numbers
 81
- extendible cardinals. 121
- extraterritoriality. 39
- factorial. 149

Index

- Factory Act of 1850. 37
- fields. 127
- first coordinate and second
 - coordinate. 93
- fourth verse of our national
 - anthem. 21
- function. 93, 178
- fundamental principle of
 - counting. 146
- generator of a group. 183
- Grand Unified Theory. 117
- group—definition. 27
- groups with four elements.
 - 95-97
- high school geometry axioms
 - 67
- inaccessible cardinals. 121
- indescribable cardinals. 121
- index set. 87
- indiscrete topology. 38
- ineffable cardinals. 121
- integers—definition. 105
- intersection. 17, 68
- isomorphic. 75-77
- Klein four-group. 97-99
- Kurt Gödel. 122
- Lagrange's theorem. 125
- law of trichotomy. 81
- least element in the natural
 - numbers. 83
- left cancellation law. 71
- left identity. 27
- left inverse. 27
- less than in the integers. 108,
 - 109
- less than in the natural numbers
 - 81
- limit point. 65
- liters vs. quarts. 16
- logs. 168
- magnetic pole reversal. 107
- Marie Antoinette. 112
- math induction. 33-35, 42
- math theory—defined. 23, 30
- metamathematics. 121, 122
- modus ponens. 18
- monoids. 126, 153
- multiplication in the natural
 - numbers. 80
- multiplication of integers. 107
- naive set theory. 17, 43
- negative times a negative. 109
- non-commutative group
 - 1st example—ice cream cones
 - 51
 - 2nd example—2 x 2 matrices
 - 54
 - 3rd example—flipping ducks
 - around. 54
- non-exclusive *or*. 17
- normal sets. 22
- null set. 16
- one-to-one (1–1) functions. 93,
 - 166
- onto. 75, 166
- open covering. 137
- open interval. 40
- open sets. 37
- ordered pair. 89-92
- partition. 101, 102
- Peano postulates. 32
- Pleiades constellation. 172
- point of accumulation. 66
- power set. 38, 146
- propositional function. 47
- \mathbb{Q} is dense in \mathbb{R} 62

Index

- range. 93, 166
- rational numbers as repeating
 decimals. 134, 155
- rational numbers \mathbb{Q} 128, 129
- reasons used in proofs. 18
- right cancellation law. 27
- rings. 126
- rows vs. columns. 165
- Schröder-Bernstein theorem
 119, 120
 proof. 193-198
- schwa. 29
- semigroups. 126, 153
- separated. 136
- set subtraction. 68
- set—defined. 45
- set-builder notation. 16, 17
- sheet music for “I Sing”. 50
- standard topology for the real
 numbers. 66
- strong induction. 83
- strongly compact cardinals. 121
- subgroup. 99, 100, 181
- subscriptsmanship. 138
- subset. 17, 70, 162
- subset axiom. 48
- subtraction in the integers. 109
- successor function. 31
- supercompact cardinals. 121
- symmetric group of degree 3
 53
- T spaces. 139
- the real numbers \mathbb{R} 130-134
 four approaches that won't
 work. 131
- Theory Of Everything. 117
- three possible meanings of (5, 8)
 40
- total binary relations. 84
- unary operation. 31
- union. 17, 162
- vacuously true. 173
- vector spaces. 127
- well-defined. 108
- Zermelo, Fraenkel, Mirimanoff,
 and Skolem. 45
- ZF #1, axiom of extensionality
 46
- ZF #2, subset axiom. 48
- ZF #3, axiom of pairing. 68
- ZF #4, axiom of union. 69
- ZF #5, the power set axiom. 89
- ZF #6, axiom of replacement
 114
- ZF #7, axiom of infinity. 116
- ZF #8, axiom of foundation
 118

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