Life of Fred®

Five Days of Upper Division Math:

Set Theory

Modern Algebra

Abstract Arithmetic

Topology

Stanley F. Schmidt, Ph.D.

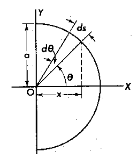
Polka Dot Publishing

A Note to Readers

I am a mathematician. That wasn't always the case. In high school the four years of math were easy compared with reading *David Copperfield* (didn't like) and *Moby Dick* (liked), typing papers (10–20 pages), or memorizing history facts. But that didn't make me a mathematician.

The first two years of college calculus were not pleasant. The teachers made us memorize stuff and gave us no real reason why we might ever use it. Opening my old calculus textbook* at random I read:

> 146. **Centroid and Moment of Inertia of Arc**
> Let the arc AB of a curve be divided into *n* parts as show in Figure 180, and let (x_k,y_k) be any point on the *k*th segment of arc Äs_k. In accordance with the definition of centroids for areas and volumes, we define the centroid of an arc as the point $(\bar{x},\bar{y})$ determined by the relations $s\bar{x}$ = lim Ó x_kÄs_k [etc.]



Taught that way, calculus was definitely not fun. My four semester grades were A, C, B, and C.**

Needless to say, I wasn't a mathematician after those two years of calculus.

At that time, there were three reasons I chose mathematics as my major: ① The grading wasn't subjective. If you got the right answer, the teacher couldn't argue. *A friend of mine was a political science major. He supported the individual over the state and 99% of the faculty were statists. On one oral Ph.D. exam he received an A and two F's. He had to change universities in order to get his doctorate.* ② Being a math major offered much better employment opportunities than any major with the word *studies* in it. ③ Math majors don't have to write long term papers.***

---

* p. 331 of Thurman S. Peterson's *Calculus with Analytic Geometry*

** Years later, when I taught college calculus, all the tests I gave were open book (no more memorizing!), and my lectures included lots of "Fred" to illustrate how calculus is relevant in everyday life. Each year I taught, I included more Fred.

*** Life is "slightly" unpredictable. This is my 35th book. Five of them have been more than 540 pages long. *Life of Fred: Trig Expanded Edition* was only 496 pages. Writing about Fred is a pure joy.

Of all my relatives, I was the first one to get a college education. No one could offer me the good news:

Once you get to
upper division pure math,
the world changes
for the better.

All the engineering majors who sat next to you in calculus have gone off to Engineering Land. They are off computing centroids of arcs and building bridges, chronometers, and skyscrapers.

For the happy few who enter the magic worlds

- of set theory
- of modern algebra
- of abstract arithmetic
- of topology

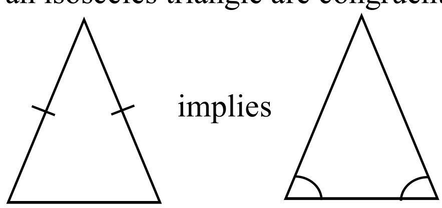
there are

- ✦ no "word problems"
- ✦ no papers to write
- ✦ no concrete applications
- ✦ no dates or formulas to memorize.

> How many ounces of Captain Mousebait cereal (7% flour) should you mix with Sergeant Sugar cereal (11% flour) to obtain 300 ounces of cereal with 8% flour? This is an actual problem, which can be found on page 28 in *Zillions of Practice Problems for Beginning Algebra.* (answer = 225 ounces.)

Instead, there are simply puzzles to solve. Often the puzzles ask the student to prove things. And just like in geometry, there can be more than one way to create a proof.

In *Life of Fred: Geometry*, we showed four different proofs that the base angles of an isosceles triangle are congruent.



One way was to draw a segment from the top vertex to the midpoint of the base (a median) and show the triangles are congruent by SSS. A second way was to draw the angle bisector from the top angle and show the triangles are congruent by SAS.

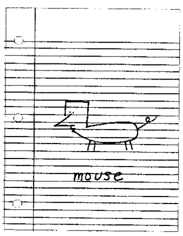
Instead of computing answers like the folks over in Engineering Land, you will be engaged in pure thought. Both activities can be hard work, but they are *different* kinds of work.

I became a mathematician during my junior and senior years at the university. I can't point to a particular instant in time when this happened, but I remember the joy of taking five math courses ☺☺☺☺☺ in my last undergraduate semester.

I have never seen any other book attempt to do what we are going to do here: The first five days in four upper division math courses taught by our master teacher Fred Gauss. Some of the puzzles (proofs) will be easy and some will be **hard**. If they were all easy, it really wouldn't be as much fun.

There will be no final exam, no grades, and no competition with other students.

I have been looking forward to writing this book for more than a decade. I love set theory, modern algebra, abstract arithmetic, and topology. One thing that has held me back is that there will be no bank robberies, no animals, no C.C. Coalback, and no boxing matches in this book. These things were easy to include when I wrote all the books from *Life of Fred: Apples* up through *Life of Fred: Calculus.* The former things are passed away. I am making all things new.*



What we'll miss

THREE PREREQUISITES FOR THIS BOOK

1. You gotta know what *prerequisite* means.
2. Two decent courses in high school algebra that included such things as unions of sets, math induction, the associative property, one-to-one functions, inverse functions, and multiplying matrices. One decent course in geometry that included lots of proofs.
3. The math in this book is the first parts of upper division mathematics for math majors. In a university setting the students in these classes are battle hardened with two years of calculus. They are used to having to work hard to understand the new material. They do not fold up into a little ball and blow away when they don't instantly understand a new concept. The third prerequisite is that you are not a fluff ball.

With my best wishes,
Stan

*I stole those two sentences from another book.

Contents

THURSDAY

FRIDAY

MONDAY
Set Theory

Prologue

It was the start of the summer classes at KITTENS University. The university president had told Fred that as long as he taught the required math courses, he was free to augment his schedule with any other classes he wished.

Fred was overjoyed. He had never been given such freedom. This was a chance to teach some junior- and senior-level math courses.

The president's secretary emailed the list of the courses he was required to teach:

8–9 Arithmetic
9–10 Beginning Algebra
10–11 Advanced Algebra
11–noon Geometry
noon–1 Trigonometry
1–2 Calculus
2–3 Statistics
3–3:05 Break
3:05–4 Linear Algebra

Fred was delighted. This was a lighter load than he had had in the spring semester. He added his four favorite upper division courses:

4–5 Set Theory
5–6 Modern Algebra
6–7 Abstract Arithmetic
7–8 Topology
8–9 Arithmetic
9–10 Beginning Algebra
10–11 Advanced Algebra
11–noon Geometry
noon–1 Trigonometry
1–2 Calculus
2–3 Statistics
3–3:05 Break
3:05–4 Linear Algebra

For a six-year-old experienced university professor like Fred, this would be a pleasant twelve-hour teaching schedule. The major difference for Fred would be that he would go jogging at 3 a.m. instead of at dawn as he had done for years.

4 a.m.

There were 300 students in the Archimedes auditorium classroom awaiting their master teacher. Two of Fred's best students, Betty and Alexander, were there.

The news had spread through the mathematics communities around the world that Fred was going to teach four upper division math courses for the first time. Many people instantly changed their summer plans and headed to KITTENS University. They filled the rest of the seats in the auditorium. The same people would be attending all four classes.

Fred entered. He had on his customary bow tie that he liked to wear when he was teaching, but had forgotten to change out of his jogging shorts. No one noticed. He waved hello and the room became silent.

Good morning. (← Fred's speech is in this font.)

This was Fred's time of day. At about 6 p.m. each evening he would be heading to bed to get his needed nine hours of sleep that every six-year-old needs. One of the students had a thermos with a liter of strong hot coffee. After he had drunk a little, he had a quart.

Some mathematicians have claimed that virtually every part of math could be ultimately based on set theory. It's a good place to start our day.

You have had high school math so you already know that a set is just any collection of things. The set containing ❀ and the number 8 can be written as {❀, 8}. Those curly parentheses are called **braces**. This is a left brace.

Fred wrote { on the blackboard.

{❀, 8} and {8, ❀} are the same set. The order in which you list the elements of the set doesn't matter.

Please don't list the same member* of a set more than once. Don't write {❀, 8, ❀}. It makes it hard to count the number of elements in a set if there are duplicates in the listing.

The **cardinality** of a set is the number of members in the set. The cardinality of the **empty set**, { }, is zero. The empty set is sometimes called the **null set** and is sometimes represented by the symbol Ø.

A second way to list a set is to use **set-builder notation**. If I wanted to list all the prime numbers that are less than a thousand, I could write {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733,

* member of a set = element of a set

739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997} or, using set-builder notation, write {x | x is a prime number less than 1000}. This is read as, "The set of all x such that x is a prime number less than 1000."

If I were to write {y | y is a prime number less than 1000} that would be the same set.

We abbreviate "is a member of" by ∈. 8 ∈ {❀, 8}

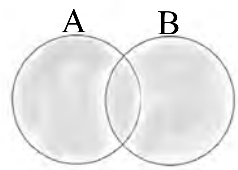
We abbreviate "is not a member of" by ∉. 9 ∉ {❀, 8}

If we have two sets, A and B, we define the **union** of A and B, A ∪ B, as {x | x ∈ A or x ∈ B}.

The **intersection** of A and B, A ∩ B, is defined as {x | x ∈ A and x ∈ B}.

Or in mathematics is the non-exclusive *or.* It means one or the other or both. Lawyers who want to indicate the non-exclusive *or* write *and/or*. Police who shout, "Stop or I'll shoot" are hopefully using the exclusive *or.* You don't want to stop *and* get shot.

The other thing we did in high school math was to draw Venn diagrams. We colored in circles.



A ∪ B

We defined **subset**:
C is a subset of D, written C ⊂ D, if every element of C was in D.

In thirteen years of school—kindergarten through 12th grade—this was set theory. In mathematics, this material is called **naive set theory**.[*]

The only thing wrong with naive set theory is that it contained contradictions. You can prove a statement is true, and you can prove its opposite is also true. Once that happens, the game is over. Everything falls apart.

In logic, if you know that statement P is true and you also know that not-P is true, then *you can prove that anything is true*. In symbols, (P & ¬P) ⇒ Q, where Q is any statement.[**]

---

[*] *Naive* is pronounced nigh-EVE. Naive = simple, unsophisticated. Coloring in circles is not really heavy-duty math.

[**] & = *and* ¬ = *not* ⇒ = *implies*

There are two famous theorems in logic. The first one is $(P \& \neg P) \Rightarrow Q$. **There shalt not be any contradictions.**

The second one is called **modus ponens**. If you know that statement P implies statement Q and you know that statement P is true, then you can infer that statement Q is also true. $(P \Rightarrow Q \;\&\; P) \Rightarrow Q$.

If your mother says, "If you do that, I'll ground you," and if you do that, then you know you'll be grounded.

Creating proofs is the heart of upper division math. One big reason you spent a year studying high school geometry was to learn how to prove things. It wasn't to learn area formulas or that the base angles of an isosceles triangle are equal. In the eighth grade you already knew that the opposite sides of a parallelogram ▱ are equal. What you learned in geometry was how to *prove* that.

The rules for doing a proof are easy: ① Every line must have a reason that justifies that line, and ② the last line must be what you want to prove. The rules for most board games are much more complicated.

Here are some of the reasons we used in geometry. We use the same ones in upper division math.

1. Given
2. Postulate or axiom. (These words mean the same thing.)
3. Definition
4. Previously proven theorem
5. Beginning of an indirect proof
6. Contradiction in steps ___ and ___, and therefore the assumption in step ___ is false.
7. Cases

Reasons 5 and 6 are always paired together. In the beginning of an indirect proof, you assume the opposite of what you want to prove. Then you derive a contradiction. That contradiction indicates that your initial assumption was false.

How to Prove You Are Alive

1. Assume I'm dead.	1. Beginning of an indirect proof
2. I couldn't be speaking to you.	2. Definition of dead
3. I am talking to you.	3.
4. I am alive.	4. Contradiction in steps 2 and 3 and therefore the assumption in step 1 is false.

If the beginning assumptions (postulates or axioms) are true, then what you prove must also be true. Mathematics is a truth-generating machine.

Of course, if the postulates are inconsistent, then the whole system crashes, and you can prove anything.

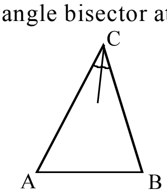
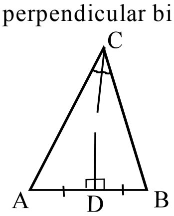
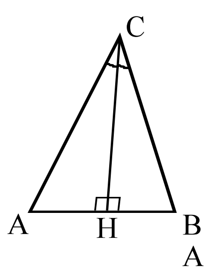
Fred giggled a little at this point.

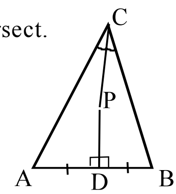
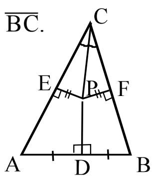
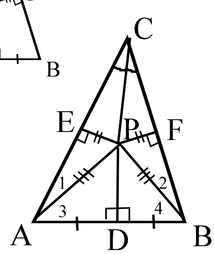
And what most geometry teachers and most geometry books fail to mention is that *The high school geometry postulates are inconsistent*.

A stunned silence fell over the audience. Everyone stopped writing and looked at Fred. Those who had not read *Life of Fred: Geometry* had no idea this was true. One of the students, Thomas, raised his hand and said, "I can't believe that. Everyone knows that high school geometry is true. Unless you show me a contradiction—one I can see and understand—"

No problem. What if I prove that every triangle is isosceles?

Thomas laughed to himself. And Fred began.

Statement	*Reason*
1. Any old triangle ABC.	1. Given
2. Draw the angle bisector at C 	2. By the angle measurement postulate $\angle C$ has a measurement between 0 and 180. And by the angle measurement postulate, there is an angle equal to half of that measurement. (Or, more simply, angle bisectors exist.)
3. Erect the perpendicular bisector of $\overline{AB}$. 	3. Theorem: Every segment has a midpoint, and Theorem: You can erect a $\perp$ to a line at any point on that line. (Or, more simply, every segment has a $\perp$ bisector.)
4. The angle bisector and the $\perp$ bisector are parallel.	4. Case 1 (One of two possibilities.)
5. The angle bisector is $\perp$ to $\overline{AB}$. 	5. Theorem: If a line (in this case $\overline{AB}$) is $\perp$ to one of two parallel lines, it is $\perp$ to the other.

6. $\angle AHC \cong \angle BHC$	6. Theorem: $\perp$ lines form congruent right angles.
7. $\overline{CH} \cong \overline{CH}$	7. Theorem: Every segment is $\cong$ to itself.
8. $\triangle AHC \cong \triangle BHC$	8. ASA
9. $\angle A \cong \angle B$	9. Definition of $\cong$ $\triangle$s.
10. $\triangle ABC$ is isosceles.	10. Converse of the Isosceles Triangle theorem. (If the base $\angle$s are $\cong$, then the $\triangle$ is isosceles.)
11. The angle bisector and the $\perp$ bisector are not parallel.	11. Case 2 (The only other possibility.)
12. They intersect. 	12. Definition of not parallel.
13. From the point of intersection, P, drop $\perp$s to $\overline{AC}$ and $\overline{BC}$. 	13. Theorem: From any point you can drop a perpendicular to a line.
14. PE = PF	14. Theorem: Any point on an $\angle$ bisector is equidistant to the sides of the angle.
15. AP = BP	15. Any point on a $\perp$ bisector is equidistant from the endpoints of the segment.
16. $\triangle APE \cong \triangle BPF$ $\triangle APD \cong \triangle BPD$ 	16. Hypotenuse-leg theorem. (In any pair of right $\triangle$s, if the hypotenuses and one pair of legs are $\cong$, then the $\triangle$s are $\cong$.)
17. $\angle 1 \cong \angle 2$ $\quad \angle 3 \cong \angle 4$	17. Definition of $\cong$ $\triangle$s.
18. $\angle CAB \cong \angle CBA$	18. Angle Addition postulate
19. $\triangle ABC$ is isosceles.	19. Converse of the Isosceles Triangle theorem. (If the base $\angle$s are $\cong$, then the $\triangle$ is isosceles.) ⊠

This marks the end of a proof. It's the same as Q.E.D.

There is nothing wrong with this proof *if* you accept the postulates of high school geometry. The problem is that high school geometry allows this to happen. The postulates allow this contradiction (and many others) to happen.

Thomas's life changed at this point. He had seen the broader vistas of upper division math.

And naive set theory contains . . .

The entire classroom said, "No. No. No. It can't be. Impossible. Incredible. No way."

. . . contradictions.

The audience looked like they had been doused with a bucket of cold water.

Even with the little bit of set theory that I've described this morning, there's enough to find a contradiction. And once you have a contradiction, $P \& \neg P$, you can prove anything. $(P \& \neg P) \Rightarrow Q$

Betty turned to Alexander and said the famous words from the "Wizard of Oz" movie: "I don't think we're in Kansas anymore."

Everyone in the audience fastened their seatbelts.* Every eye was on Fred.

First of all, let's consider all those sets that are members of themselves.** Let A be the name of this set.

Then $A = \{x \mid x \in x\}$.

Is $A \in A$?

Obviously yes. It satisfies the definition: $\{x \mid x \in x\}$.

Most sets do *not* contain themselves as members. For example, the set of natural numbers, $\{1, 2, 3, 4, 5, \ldots\}$. Or the set of all ducks.

Or the set of all fish that know the words to the fourth verse of our national anthem. Another name for this last set is the empty set, { }, or Ø.

> Oh! thus be it ever, when freemen shall stand
> Between their loved homes and the war's desolation!
> Blest with victory and peace, may the
>   heaven-rescued land
> Praise the Power that hath made and preserved us a
>   nation.
> Then conquer we must, when our cause it is just,
> And this be our motto: "In God is our trust."
> And the star-spangled banner in triumph shall wave
> O'er the land of the free and the home of the brave!

---

* All of the classrooms that Fred teaches in have seatbelts. Boring teachers should have classrooms with pillows.

** Sets that contain themselves as members are fairly rare. Most sets do not contain themselves as a member. The set of your hands contains  . It doesn't contain any sets inside the braces.

One set that is a member of itself is the set of all sets mentioned in this book.

Sets that do not contain themselves as members are called **normal sets**. Let's let B equal the set of all normal sets. $B = \{x \mid x \notin x\}$

Our hour is almost up. For Tuesday please create proofs for these two theorems.*

Theorem 1: If $B \in B$, then $B \notin B$.
Theorem 2: If $B \notin B$, then $B \in B$.

After you have proven both of these, you have established that

$$B \in B \text{ iff } B \notin B.$$

(iff = if and only if)

This is a contradiction.

Note to readers:
Answers to all of Fred's assignments are given in the back of this book.
Please do not just read the question and just turn to the answer. You won't learn very much if you do that.
Write out your answers first.
The fun part of this upper division math is solving the puzzles—not just learning "stuff."

On Tuesday we will go beyond naive set theory.
I will present a list of set theory axioms that do not contain any contradictions.

I'll see you tomorrow.

* A theorem is a statement that has been or can be proven.

Index

Index