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STUDENT

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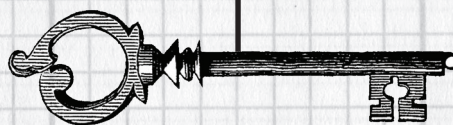
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# INTERMEDIATE LOGIC

*Mastering Propositional Arguments*

STUDENT: THIRD EDITION

Canon Logic Series





# INTERMEDIATE LOGIC

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## CONTENTS

Preface to the First Edition . . . . .	1
Preface to the Second Edition . . . . .	2
Preface to the Third Edition . . . . .	3
Introduction . . . . .	5

### UNIT ONE: TRUTH TABLES

Lesson 1: Introduction to Propositional Logic . . . . .	9
<i>Exercise 1</i> . . . . .	13
Lesson 2: Negation, Conjunction, and Disjunction . . . . .	15
<i>Exercise 2</i> . . . . .	19
Lesson 3: Truth Tables for Determining Truth Values . . . . .	21
<i>Exercise 3</i> . . . . .	25
Lesson 4: The Conditional . . . . .	27
<i>Exercise 4</i> . . . . .	33
Lesson 5: The Biconditional . . . . .	35
<i>Exercise 5</i> . . . . .	37
Lesson 6: Logical Equivalence and Contradiction . . . . .	39
<i>Exercise 6</i> . . . . .	41
Lesson 7: Truth Tables for Determining Validity . . . . .	43
<i>Exercise 7a</i> . . . . .	47
<i>Exercise 7b</i> . . . . .	51
Lesson 8: Shorter Truth Tables for Determining Validity . . . . .	53
<i>Exercise 8</i> . . . . .	57
Lesson 9: Using Assumed Truth Values in Shorter Truth Tables . . . . .	59
<i>Exercise 9</i> . . . . .	63
Lesson 10: Shorter Truth Tables for Consistency . . . . .	65
<i>Exercise 10</i> . . . . .	67
Lesson 11: Shorter Truth Tables for Equivalence . . . . .	69
<i>Exercise 11</i> . . . . .	71
Lesson 12: The Dilemma . . . . .	73
<i>Exercise 12</i> . . . . .	77
Unit 1 Review Questions . . . . .	79
Unit 1 Review Exercises . . . . .	83

## UNIT TWO: FORMAL PROOFS OF VALIDITY

Lesson 13: The Rules of Inference . . . . .	101
<i>Exercise 13</i> . . . . .	107
Lesson 14: Recognizing the Rules of Inference . . . . .	109
<i>Exercise 14a</i> . . . . .	113
<i>Exercise 14b</i> . . . . .	115
Lesson 15: Developing Formal Proofs . . . . .	117
<i>Exercise 15a</i> . . . . .	119
<i>Exercise 15b</i> . . . . .	121
Lesson 16: The Rules of Replacement . . . . .	123
<i>Exercise 16</i> . . . . .	127
Lesson 17: Practice with Proofs . . . . .	129
<i>Exercise 17a</i> . . . . .	133
<i>Exercise 17b</i> . . . . .	135
Lesson 18: The Conditional Proof . . . . .	137
<i>Exercise 18</i> . . . . .	141
Lesson 19: Reductio ad Absurdum . . . . .	143
<i>Exercise 19</i> . . . . .	147
Lesson 20: Proving Rules Unnecessary . . . . .	149
<i>Exercise 20</i> . . . . .	151
Lesson 21: Truth-Functional Completeness . . . . .	153
<i>Exercise 21</i> . . . . .	159
Unit 2 Review Questions . . . . .	161
Unit 2 Review Exercises . . . . .	163

## UNIT THREE: TRUTH TREES

Lesson 22: Truth Trees for Consistency . . . . .	177
<i>Exercise 22</i> . . . . .	183
Lesson 23: Decomposition Rules . . . . .	185
<i>Exercise 23</i> . . . . .	191
Lesson 24: Techniques for Constructing Truth Trees . . . . .	193
<i>Exercise 24</i> . . . . .	197
Lesson 25: Truth Trees for Self-Contradiction and Tautology . . . . .	199
<i>Exercise 25</i> . . . . .	203
Lesson 26: Truth Trees for Equivalence . . . . .	205
<i>Exercise 26</i> . . . . .	207
Lesson 27: Truth Trees for Validity . . . . .	209
<i>Exercise 27</i> . . . . .	213
Unit 3 Review Questions . . . . .	215
Unit 3 Review Exercises . . . . .	217

## UNIT FOUR: APPLYING THE TOOLS TO ARGUMENTS

Lesson 28: Applying the Tools to Actual Arguments . . . . .	229
<i>Exercise 28a</i> . . . . .	233
<i>Exercise 28b</i> . . . . .	235
<i>Exercise 28c</i> . . . . .	237
Unit 4 Review Questions . . . . .	241
Unit 4 Review Exercises . . . . .	243

## UNIT FIVE: DIGITAL LOGIC

Lesson 29: Digital Displays . . . . .	249
<i>Exercise 29</i> . . . . .	251
Lesson 30: Numbers in Different Bases . . . . .	253
<i>Exercise 31</i> . . . . .	257
Lesson 31: Binary Arithmetic . . . . .	259
<i>Exercise 31</i> . . . . .	263
Lesson 32: Basic Logic Gates . . . . .	265
<i>Exercise 32</i> . . . . .	269
Lesson 33: Finding Truth Tables from Digital Logic Circuits . . . . .	271
<i>Exercise 33</i> . . . . .	275
Lesson 34: Converting Truth Tables into Digital Logic Circuits . . . . .	277
<i>Exercise 34</i> . . . . .	281
Lesson 35: NAND Gates and NOR Gates . . . . .	283
<i>Exercise 35</i> . . . . .	289
Lesson 36: Simplification Rules . . . . .	291
<i>Exercise 36</i> . . . . .	295
Lesson 37: Rules of Replacement . . . . .	297
<i>Exercise 37</i> . . . . .	301
Lesson 38: De Morgan's Theorem . . . . .	303
<i>Exercise 38</i> . . . . .	305
Lesson 39: Two More Logic Gates . . . . .	307
<i>Exercise 39</i> . . . . .	311
Lesson 40: Karnaugh Mapping . . . . .	313
<i>Exercise 40a</i> . . . . .	321
<i>Exercise 40b</i> . . . . .	323
Unit 5 Review Questions . . . . .	327
Unit 5 Review Exercises . . . . .	331

## APPENDICES

Appendix A: Defining Truth Tables for Five Logical Operators . . . . .	351
Appendix B: Rules of Inference and Replacement . . . . .	353
Appendix C: Truth Tree Decomposition Rules . . . . .	355
Appendix D: Digital Logic Simplification Rules . . . . .	357





## PREFACE to the FIRST EDITION

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This text, *Intermediate Logic: Mastering Propositional Arguments*, is designed as a continuation to *Introductory Logic: The Fundamentals of Thinking Well*. Together, these two textbooks should provide sufficient material for a complete course in basic logic.

We have attempted to make this a useable workbook for the logic student. To that end we have included exercises for every lesson, each of which has been developed and used over many years of logic classes. The goal is to keep the text clear and complete, such that an adult could teach himself the fundamentals of logic.

A number of other logic texts were consulted throughout the writing of *Intermediate Logic*. Most helpful was Irving Copi's invaluable *Introduction To Logic* (Macmillan Publishing Co., 1978). While we did not lift material directly from it, of course, that book has so shaped our own understanding of this subject that *Intermediate Logic* undoubtedly echoes much of its format and contents. *Intermediate Logic* has also benefitted from *The Art of Reasoning* by David Kelley (W. W. Norton & Company, Inc., 1990) and *The Logic Book* by Bergmann, Moor, and Nelson (McGraw-Hill, Inc., 1990).

Although we cannot list them here, we are indebted to many people for the completion of this project. We give special credit to the students throughout the years who have been introduced to the beauty and practicality of the world of logic found in the pages of these textbooks. Good students always force teachers to re-evaluate their own understanding of a subject, and such students have contributed more to this book than we or they realize.

## PREFACE to the SECOND EDITION

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The subject of logic may be divided into two main branches: formal and informal. The definition of logic as “the science and the art of correct reasoning” allows us to distinguish these two branches. Formal logic deals *directly* with reasoning. Reasoning means “drawing conclusions from other information.” Whenever we consider how to analyze and write logical arguments—in which conclusions are drawn from premises—we are working in the realm of formal logic. Informal logic, on the other hand, deals more *indirectly* with reasoning. When we argue, we often find ourselves defining terms, determining the truth values of statements, and detecting spurious informal fallacies. While in none of these activities do we concentrate on reasoning in a formal way, we do recognize that such activities are indirectly related to and support the process of reasoning, and are thus best included under informal logic.

With this in mind, several changes were made in 2006 to this second edition of *Intermediate Logic*. If it’s of interest, those changes are listed below.

First, in order to present to the student a more logical progression of topics, the section on defining terms from the first edition has been entirely removed from this text and placed at the beginning of *Introductory Logic*, where it is taught along with other branches of informal logic and categorical logic. Consequently, this text now focuses solely on the branch of formal logic called propositional logic, of which formal proofs of validity and truth trees are subsets.

Second, review questions and review exercises have been added to each unit for every lesson in the text, effectively doubling the number of exercises for students to verify their knowledge and develop their understanding of the material. Additionally, some especially challenging problems which relate to the material have been included in the review exercises. Students who can correctly answer all of the review questions demonstrate a sufficient knowledge of the important concepts. Students who can correctly solve the review exercises demonstrate a sufficient understanding of how to apply those concepts.

Third, the definitions of important terms, key points made, and caution signs regarding common errors are now set apart in the margins of the text. This should help students to distinguish the most important topics, as well as aid in their review of the material.

Fourth, every lesson has been reviewed in detail with the goal of improving the clarity of the explanations and correcting several minor errors that were found in the first edition. To all logic students and teachers goes the credit for any improvements that have been made in this second edition; for those remaining errors and defects we take full responsibility.

## PREFACE to the THIRD EDITION

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Formal logic is a fascinating subject. Students are often intrigued by the concepts and methods of reasoning revealed in it. Truth tables, formal proofs, and other operations of propositional logic challenge their ability to think abstractly, and provide opportunities to practice and develop their puzzle solving skills. Pure logic is fun. But for students to learn how to reason properly, and through the process of reasoning to recognize and discover truth, they must learn how to apply these methods of formal logic to the world around them.

Many teachers and parents want logic to be practical. They want their logic students to be able to employ symbolic logic as a tool of thinking, a tool both powerful and flexible enough to use in many different ways and on many different media.

With these things in mind, in 2014 we added two important sections on the practical applications of propositional logic to the third edition.

First, a new lesson teaches students how to apply the tools that they have learned to actual arguments. Lesson 28 teaches students how to analyze chains of reasoning found in writings such as philosophy and theology. Though some new concepts are introduced, most of this lesson is aimed at teaching students to employ what they have learned in the previous 27 lessons. This one lesson therefore has three corresponding exercises, giving students the opportunity to work through small portions of three ancient texts: Boethius's *The Consolation of Philosophy*, the Apostle Paul's argument on the resurrection from 1 Corinthians 15, and a section on angelic will from Augustine's *City of God*. The additional exercises for this lesson also consider Deuteronomy 22, a portion from Martin Luther's sermon on John chapter 1, and a witty interchange that you may have seen in the 2008 action comedy *Get Smart*.

Second, a longer optional unit has been added on the useful and stimulating topic of digital logic. Unit 5 includes twelve new lessons that unlock the logic of electronic devices. These lessons work through the concepts of digital displays, binary numbers, and the design and simplification of digital logic circuits. Many of the lessons learned earlier in the text are given new and intriguing applications, as students learn how to employ propositional logic to understand the electronic gadgets that they see and use every day. The final exercise gives students the opportunity to design a complex circuit that can convert a binary input to a decimal display output. This unit has become a favorite of many logic students over the past decades.

It is our hope that these additions give students a vision of the power of propositional logic and fulfill teachers' and parents' desires to make propositional logic more practical.



## INTRODUCTION

Logic has been defined both as the *science* and the *art* of correct reasoning. People who study different sciences observe a variety of things: biologists observe living organisms, astronomers observe the heavens, and so on. From their observations they seek to discover natural laws by which God governs His creation. The person who studies logic as a science observes the mind as it reasons—as it draws conclusions from premises—and from those observations discovers laws of reasoning which God has placed in the minds of people. Specifically, he seeks to discover the principles or laws which may be used to distinguish good reasoning from poor reasoning. In deductive logic, good reasoning is *valid* reasoning—in which the conclusions follow necessarily from the premises. Logic as a science discovers the principles of valid and invalid reasoning.

Logic as an *art* provides the student of this art with practical skills to construct arguments correctly as he writes, discusses, debates, and communicates. As an art logic also provides him with rules to judge what is spoken or written, in order to determine the validity of what he hears and reads. Logic as a science discovers rules. Logic as an art teaches us to apply those rules.

Logic may also be considered as a symbolic language which represents the reasoning inherent in other languages. It does so by breaking the language of arguments down into symbolic form, simplifying them such that the arrangement of the language, and thus the reasoning within it, becomes apparent. The outside, extraneous parts of arguments are removed like a biology student in the dissection lab removes the skin, muscles and organs of a frog, revealing the skeleton of bare reasoning inside. Thus revealed, the logical structure of an argument can be examined, judged and, if need be, corrected, using the rules of logic.

So logic is a symbolic language into which arguments in other languages may be translated. Now arguments are made up of propositions, which in turn are made up of terms. In categorical logic, symbols (usually capital letters) are used to represent terms. Thus “All men are sinners” is translated “All M are S.” In propositional logic, the branch of logic with which this book primarily deals, letters are used to represent entire propositions. Other symbols are used to represent the logical operators which modify or relate those propositions. So the argument, “If I don’t eat, then I will be hungry; I am not hungry, so I must have eaten” may appear as  $\sim E \supset H, \sim H, \therefore E$ .

Unit 1 of this book covers the translation and analysis of such propositional arguments, with the primary concern of determining the validity of those arguments. Unit 2 introduces a new kind of logical exercise: the writing of formal proofs of validity and related topics. Unit 3 completes propositional logic with a new technique for analyzing arguments: truth trees. Unit 4 considers how to apply these tools and techniques to arguments contained in real-life writings: philosophy, theology, and the Bible itself. Unit 5 introduces digital logic and helps students to unlock the logic of electronic devices.





## UNIT 1

# TRUTH TABLES

## CONTENTS

Lesson 1: Introduction to Propositional Logic . . . . .	9
<i>Exercise 1</i> . . . . .	13
Lesson 2: Negation, Conjunction, and Disjunction . . . . .	15
<i>Exercise 2</i> . . . . .	19
Lesson 3: Truth Tables for Determining Truth Values . . . . .	21
<i>Exercise 3</i> . . . . .	25
Lesson 4: The Conditional . . . . .	27
<i>Exercise 4</i> . . . . .	33
Lesson 5: The Biconditional . . . . .	35
<i>Exercise 5</i> . . . . .	37
Lesson 6: Logical Equivalence and Contradiction . . . . .	39
<i>Exercise 6</i> . . . . .	41
Lesson 7: Truth Tables for Determining Validity . . . . .	43
<i>Exercise 7a</i> . . . . .	47
<i>Exercise 7b</i> . . . . .	51
Lesson 8: Shorter Truth Tables for Determining Validity . . . . .	53
<i>Exercise 8</i> . . . . .	57
Lesson 9: Using Assumed Truth Values in Shorter Truth Tables . . . . .	59
<i>Exercise 9</i> . . . . .	63
Lesson 10: Shorter Truth Tables for Consistency . . . . .	65
<i>Exercise 10</i> . . . . .	67
Lesson 11: Shorter Truth Tables for Equivalence . . . . .	69
<i>Exercise 11</i> . . . . .	71
Lesson 12: The Dilemma . . . . .	73
<i>Exercise 12</i> . . . . .	77
Unit 1 Review Questions . . . . .	79
Unit 1 Review Exercises . . . . .	83





# INTRODUCTION TO PROPOSITIONAL LOGIC

Propositional logic is a branch of formal, deductive logic in which the basic unit of thought is the proposition. A **proposition** is a statement, a sentence which has a truth value. A single proposition can be expressed by many different sentences. The following sentences all represent the same proposition:

*God loves the world.*  
*The world is loved by God.*  
*Deus mundum amat.*

These sentences represent the same proposition because they all have the same meaning.

In propositional logic, letters are used as symbols to represent propositions. Other symbols are used to represent words which modify or combine propositions. Because so many symbols are used, propositional logic has also been called “symbolic logic.” Symbolic logic deals with **truth-functional propositions**. A proposition is truth-functional when the truth value of the proposition depends upon the truth value of its component parts. If it has only one component part, it is a **simple proposition**. A categorical statement is a simple proposition. The proposition *God loves the world* is simple. If a proposition has more than one component part (or is modified in some other way), it is a **compound proposition**. Words which combine or modify simple propositions in order to form compound propositions (words such as *and* and *or*) are called **logical operators**.

For example, the proposition *God loves the world and God sent His Son* is a truth-functional, compound proposition. The word *and* is the logical operator. It is truth functional because its truth value depends upon the truth value of the two simple propositions which make it up. It is in fact a true proposition, since it is true that God



## DEFINITIONS

**Propositional logic** is a branch of formal, deductive logic in which the basic unit of thought is the proposition. A **proposition** is a statement.



## KEY POINT

One proposition may be expressed by many different sentences.



## DEFINITIONS

A proposition is **truth-functional** when its truth value depends upon the truth values of its component parts.

If a proposition has only one component part, it is a **simple proposition**. Otherwise, it is **compound**.



## DEFINITIONS

**Logical operators** are words that combine or modify simple propositions to make compound propositions.

A **propositional constant** is an uppercase letter that represents a single, given proposition.

A **propositional variable** is a lowercase letter that represents any proposition.

loves the world, and it is true that God sent His Son. Similarly, the proposition *It is false that God loves the world* is compound, the phrase *it is false that* being the logical operator. This proposition is also truth-functional, depending upon the truth value of the component *God loves the world* for its total truth value. If *God loves the world* is false, then the proposition *It is false that God loves the world* is true, and vice versa.

However, the proposition *Joe believes that God loves the world*, though compound (being modified by the phrase *Joe believes that*), is *not* truth-functional, because its truth value does not depend upon the truth value of the component part *God loves the world*. The proposition *Joe believes that God loves the world* is a self-report and can thus be considered true, regardless of whether or not *God loves the world* is true.

When a given proposition is analyzed as part of a compound proposition or argument, it is usually abbreviated by a capital letter, called a **propositional constant**. Propositional constants commonly have some connection with the propositions they symbolize, such as being the first letter of the first word, or some other distinctive word within the proposition. For example, the proposition *The mouse ran up the clock* could be abbreviated by the propositional constant M. On the other hand, *The mouse did not run up the clock* may be abbreviated  $\sim M$  (read as *not M*). Within one compound proposition or argument, the same propositional constant should be used to represent a given proposition. Note that a simple proposition cannot be represented by more than one constant.

When the *form* of a compound proposition or argument is being emphasized, we use **propositional variables**. It is customary to use lowercase letters as propositional variables, starting with the letter *p* and continuing through the alphabet (*q, r, s, . . .*). Whereas a propositional constant represents a single, given proposition, a propositional variable represents an unlimited number of propositions.

It is important to realize that a single constant or variable can represent not only a simple proposition but also a compound proposition. The variable *p* could represent *God loves the world* or it could represent *God loves the world but He hates sin*. The entire compound



## KEY POINT

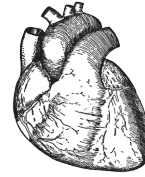
A propositional constant or variable can represent a simple proposition or a compound proposition.

proposition *It is false that if the mouse ran up the clock, then, if the clock did not strike one, then the mouse would not run down* could be abbreviated by a single constant F, or it could be represented by symbolizing each part, such as  $\sim(M \supset (\sim S \supset \sim D))$ . The decision concerning how to abbreviate a compound proposition depends on the purpose for abbreviating it. We will learn how to abbreviate compound propositions in the next few lessons.

---

**SUMMARY**

A proposition is a statement. Propositions are truth-functional when the truth value of the proposition depends upon the truth value of its component parts. Propositions are either simple or compound. They are compound if they are modified or combined with other propositions by means of logical operators. Propositional constants are capital letters which represent a single given proposition. Propositional variables are lower case letters which represent an unlimited number of propositions.





**EXERCISE 1** (25 points)

What are two main differences between propositional constants and propositional variables?

1. \_\_\_\_\_
2. \_\_\_\_\_

Modify or add to the simple proposition *We have seen God* to create the following:

3. A truth-functional compound proposition:

\_\_\_\_\_

4. A proposition which is *not* truth-functional:

\_\_\_\_\_

Circle S if the given proposition is simple. Circle C if it is compound.

- |   |   |   |
|---|---|---|
| 5. The Lord will cause your enemies to be defeated before your eyes.                  | S | C |
| 6. There is a way that seems right to a man but in the end it leads to death.         | S | C |
| 7. The fear of the Lord is the beginning of wisdom.                                   | S | C |
| 8. If we confess our sins then He is faithful to forgive us our sins.                 | S | C |
| 9. It is false that a good tree bears bad fruit and that a bad tree bears good fruit. | S | C |
| 10. The Kingdom of God is not a matter of talk but of power.                          | S | C |

Given that      B means *The boys are bad*      M means *The man is mad*  
                      G means *The girls are glad*      S means *The students are sad*

Translate the following compound propositions:

11. It is false that B. \_\_\_\_\_
  12. B or G. \_\_\_\_\_
  13. B and M. \_\_\_\_\_
  14. If M then S. \_\_\_\_\_
  15. If not M and not S then G. \_\_\_\_\_
- \_\_\_\_\_



# NEGATION, CONJUNCTION, AND DISJUNCTION

We will begin our study of abbreviating and analyzing compound propositions by learning about three fundamental logical operators: *negation*, *conjunction*, and *disjunction*. As we do, we will be answering three questions for each logical operator: What words in English are abbreviated by it? What is its symbol? How is the truth value of the compound proposition affected by the truth values of the component parts?

## Negation

Negation is the logical operator representing the words *not*, *it is false that*, or any other phrase which denies or contradicts the proposition. As we have already seen, the symbol  $\sim$  (called a *tilde*) represents negation. If the proposition *All roads lead to Rome* is represented by the propositional constant  $R$ , then  $\sim R$  means *Not all roads lead to Rome* or *It is false that all roads lead to Rome*. Note that the negation of a proposition is the contradiction of that proposition. Thus  $\sim R$  could also be translated *Some roads do not lead to Rome*. If a proposition is true, its negation is false. If a proposition is false, its negation is true. This can be expressed by the following **truth table**, where T means *true* and F means *false*:

p	$\sim p$
T	F
F	T

Truth tables show how the truth value of a compound proposition is affected by the truth value of its component parts. The table above is called the **defining truth table** for negation because it completely defines its operations on a minimum number of variables (in this case, one). The defining truth table for an operator that joins two propositions would require two variables.



## KEY POINT

Three fundamental logical operators are **negation**, **conjunction**, and **disjunction**.



## DEFINITIONS

**Negation** ( $\sim$ , *not*) is the logical operator that denies or contradicts a proposition.

A **truth table** is a listing of the possible truth values for a set of one or more propositions. A **defining truth table** displays the truth values produced by a logical operator modifying a minimum number of variables.



## DEFINITIONS

**Conjunction** ( $\bullet$ , *and*) is a logical operator that joins two propositions and is true if and only if both the propositions (*conjuncts*) are true.

## Conjunction

When two propositions are joined by *and*, *but*, *still*, or other similar words, a **conjunction** is formed. The conjunction logical operator is symbolized by  $\bullet$  (called, of course, a *dot*). If *Main Street leads to home* is represented by the constant  $H$ , then *All roads lead to Rome, but Main Street leads to home* could be represented by  $R \bullet H$  (read as *R dot H*, or *R and H*).

The conjunction is true if and only if its components (called **conjuncts**) are both true. If either conjunct is false, the conjunction as a whole is false. The defining truth table for conjunction is therefore:

p	q	$p \bullet q$
T	T	T
T	F	F
F	T	F
F	F	F

Thus if *All roads lead to Rome* is false and *Main Street leads to home* is true, then the entire conjunction *All roads lead to Rome but Main Street leads to home* is false, as seen on the third row down.

In ordinary English, the conjunction is not always placed between two distinct sentences. For example, *Paul and Apollos were apostles* could be symbolized  $P \bullet A$ , where  $P$  means *Paul was an apostle* and  $A$  means *Apollos was an apostle*. Similarly, the proposition *Jesus is both God and man* could be represented by  $G \bullet M$ .



## DEFINITIONS

**Disjunction** ( $\vee$ , *or*) is a logical operator that joins two propositions and is true if and only if one or both of the propositions (*disjuncts*) is true.

## Disjunction

A **disjunction** is formed when two propositions are joined by the logical operator *or*, as in *Paul was an apostle or Apollos was an apostle*. The symbol for disjunction is  $\vee$  (called a *vee*). The foregoing disjunction would thus be symbolized  $P \vee A$  (read simply *P or A*).

In English, the word *or* is ambiguous. In one sense it can mean “this or that, but not both” (called the *exclusive or*). For example, in the sentence *The senator is a believer or an unbeliever*, the word *or* must be taken in the exclusive sense; nobody could be both a believer and an unbeliever at the same time in the same way. However, the word *or* can also mean “this or that, or both” (called the *inclusive or*). This is how it should be taken in the sentence *Discounts are given either*



to senior citizens or war veterans. If you were a senior citizen or a war veteran or both, you would be allowed a discount.

In Latin, the ambiguity is taken care of by two separate words: *aut*, meaning the “exclusive or,” and *vel*, meaning the “inclusive or.” Although it may seem like the exclusive sense of the word *or* is the more natural sense, in logic the disjunction is always taken in the inclusive sense. This is seen in the fact that the symbol  $\vee$  is derived from the Latin *vel*.

The defining truth table for disjunction is therefore:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

A disjunction is thus considered to be false if and only if both components (called **disjuncts**) are false. If either disjunct is true, the disjunction as a whole is true.

If the context of an argument requires that the word *or* be represented in the exclusive sense, as in *The senator is either a Republican or a Democrat*, it may be translated with the more complicated  $(R \vee D) \cdot \sim(R \cdot D)$ —that is, “The senator is either a Republican or a Democrat, but not both a Republican and a Democrat.” However, you should assume that *or* is meant in the more simple inclusive sense unless instructed otherwise.

As you can see, logic may use parentheses in symbolizing complicated compound propositions. This is done to avoid ambiguity. The compound proposition  $A \vee B \cdot C$  could mean *A or B, and C* or it could mean *A, or B and C*. Parentheses remove the ambiguity, as in  $(A \vee B) \cdot C$ , which represents *A or B, and C*. This is similar to how parentheses are used in mathematics. Assuming there are no rules about which operation should be performed first, the mathematical expression  $5 + 6 \times 4$  could equal either 44 or 29, depending on whether one adds first or multiplies first. But parentheses would make it clear, as in  $(5 + 6) \times 4$ . Logic uses parentheses in the same way. Generally, in a series of three or more connected propositions, parentheses should be used.



### KEY POINT

The logical operator for disjunction is always understood in the inclusive sense: “this or that, or both.” If you intend the exclusive *or*, you must specify it explicitly.



### CAUTION

Though in English grammar the word *or* is called a conjunction, in logic only *and* (and equivalent words) is a conjunction. *Or* is always called a disjunction.



### KEY POINT

Generally, in a series of three or more connected propositions, parentheses should be used to avoid ambiguity.



### CAUTION

Do not confuse the propositional meaning of the phrases *not both* and *both not*. Use parentheses to distinguish between them.



### KEY POINT

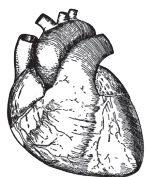
In the absence of parentheses, assume that negation attaches only to the proposition it immediately precedes.

The word *both* is often an indicator of how parentheses are to be placed when using conjunctions. The symbolized *exclusive or* in the paragraph above could be read *R or D, but not both R and D*, the word *both* telling us to place parentheses around  $R \bullet D$ .

A proper use of parentheses can also help us to distinguish between *not both* and *both not* propositions. For example, the proposition *Cats and snakes are not both mammals* (which is true) would be symbolized as  $\sim(C \bullet S)$ . The *not* comes before the *both*, so the tilde is placed before the parenthesis. However, *Both cats and snakes are not mammals* (which is false) would be symbolized as  $(\sim C \bullet \sim S)$ . Note that this second proposition could also be translated *Neither cats nor snakes are mammals*.

When symbolizing compound propositions which use negation, it is standard practice to assume that whatever variable, constant, or proposition in parentheses the tilde immediately precedes is the one negated. For example, the compound proposition  $\sim p \vee q$  is understood to mean  $(\sim p) \vee q$ , because the tilde immediately precedes the variable  $p$ . This is different from  $\sim(p \vee q)$ . Negation is used in the same way that the negative sign is used in mathematics. The mathematical expression  $-5 + 6$  means  $(-5) + 6$ , which equals  $1$ . This is different from  $-(5 + 6)$ , which equals  $-11$ . So when negating a single variable or constant, you need not use parentheses. But when negating an entire compound proposition, place the tilde in front of the parentheses around the proposition.

### SUMMARY



Three common logical operators are negation (*not*, symbolized  $\sim$ ), conjunction (*and*, symbolized  $\bullet$ ), and disjunction (*or*, symbolized  $\vee$ ). These logical operators can be defined by means of truth tables. Negation reverses the truth value of a proposition, conjunction is true if and only if both conjuncts are true, and disjunction is false if and only if both disjuncts are false.

**EXERCISE 2** (26 points)

Given: J means *Joseph went to Egypt*      F means *There was a famine*  
           I means *Israel went to Egypt*        S means *The sons of Israel became slaves*

Translate the symbolic propositions.

1.  $F \bullet I$  \_\_\_\_\_
2.  $\sim J \vee S$  \_\_\_\_\_
3.  $\sim(J \vee I)$  \_\_\_\_\_
4.  $J \bullet \sim S$  \_\_\_\_\_

Symbolize the compound propositions.

5. Joseph and Israel went to Egypt. \_\_\_\_\_
6. Israel did not go to Egypt. \_\_\_\_\_
7. Israel went to Egypt, but his sons became slaves. \_\_\_\_\_
8. Either Joseph went to Egypt, or there was a famine. \_\_\_\_\_
9. Joseph and Israel did not both go to Egypt. \_\_\_\_\_
10. Neither Joseph nor Israel went to Egypt. \_\_\_\_\_
11. Joseph and Israel went to Egypt; however, there was a famine, and the sons of Israel became slaves. \_\_\_\_\_
12. Israel went to Egypt; but either Joseph did not go to Egypt, or there was a famine. \_\_\_\_\_



# TRUTH TABLES FOR DETERMINING TRUTH VALUES

So far we have seen that truth tables help define logical operators. Truth tables also serve other functions, one of which is to help us determine the truth value of compound propositions. The truth value of elementary negations, conjunctions and disjunctions can be immediately determined from their defining truth tables. But what about compound propositions like  $\sim p \vee (\sim q \cdot r)$ ? To find the truth values for such complicated propositions, the following procedure may be followed:

1. Draw a line, and on the leftmost part of the line place the variables (or constants) which are used in the proposition. Under these, put all the possible combinations of true and false. This will require four rows for two variables, eight rows for three variables, and in general  $2^n$  rows for  $n$  variables. Under the first variable, place a T for each of the first half of the rows, then an F for each of the second half. Under the next variable, place half again as many Ts, half again as many Fs, then repeat this. The final column should have alternating single Ts and Fs, as follows:

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

You can verify for yourself that all the possible combinations of true and false are found in these eight rows.



## KEY POINT

Truth tables help determine the truth value of compound propositions, given the truth value of their component parts.



### KEY POINT

When completing a truth table, start with the standard truth values for the variables (or constants), then find the truth values for the negated variables (or constants).

2. If any variables are negated, these should be added next, with the corresponding truth values under them (specifically, under the operator):

$\downarrow$	$\downarrow$			
p	q	r	$\sim p$	$\sim q$
T	T	T	F	F
T	T	F	F	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	T
F	F	F	T	T

Here arrows are placed over  $p$  and  $q$  to show that those basic variables are being used to build more complicated propositions on the right-hand side of the table. Whenever  $p$  is true,  $\sim p$  is false, and vice versa, just as the defining truth table for negation shows. This is also the case for  $q$  and  $\sim q$ .



### KEY POINT

After determining truth values for negations, complete the truth values for compound propositions within parentheses.

3. Continue to the next level of complexity in the proposition. As in mathematics, whatever is in parentheses should be completed before going outside the parentheses. In our example, the proposition in parentheses is  $\sim q \cdot r$ . This is placed on the line, and whenever both  $\sim q$  and  $r$  are true, the conjunction  $\sim q \cdot r$  is true, according to the defining truth table for conjunction. Thus we now have:

		$\downarrow$		$\downarrow$	
p	q	r	$\sim p$	$\sim q$	$(\sim q \cdot r)$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	F

4. Continue with the same procedure, adding on to the truth table until the entire compound proposition is filled out. In our example, the propositions  $\sim p$  and  $(\sim q \cdot r)$  are disjuncts. Thus, whenever

either is true, the whole disjunction is true. We fill in those values and finish the truth table:

p	q	r	↓ ~p	↓ ~q	↓ (~q • r)	~p ∨ (~q • r)
T	T	T	F	F	F	F
T	T	F	F	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	F	F
F	T	T	T	F	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

We see from the first row that whenever  $p$ ,  $q$ , and  $r$  are all true, the compound proposition  $\sim p \vee (\sim q \cdot r)$  is false, and so on down the truth table. As you get more familiar with this procedure, you will be able to dispense with the initial guide columns of true and false, working only with the compound proposition and placing the truth values directly beneath the variables in it.

Sometimes, the truth values of constants in a compound proposition are already known. In that case finding the truth value of the compound proposition requires only one row. For instance, assume that  $A$  is true, and  $X$  and  $Y$  are false. Finding the truth value of  $(A \vee X) \cdot \sim Y$  requires this:

A	X	Y	(A ∨ X)	~Y	(A ∨ X) • ~Y
T	F	F	T	T	T



### KEY POINT

Finish the truth table for a compound proposition by finding out the truth tables for all of its component parts and then putting them together.

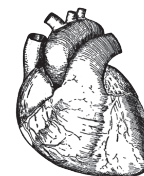


### KEY POINT

When you make a truth table for propositions that use only constants with known truth values, you need just one row.

### SUMMARY

The truth values of a compound proposition may be determined by placing all possible combinations of true and false under the variables or constants, then using the definitions of the logical operators to determine the corresponding truth values of each component of the proposition.







**EXERCISE 3** (26 points)

1. Fill in the following truth table to determine the truth values for the exclusive *or*. The truth values for  $p$  and  $q$  are filled out for you on this first one.

$p$	$q$	$(p \vee q)$	$(p \bullet q)$	$\sim(p \bullet q)$	$(p \vee q) \bullet \sim(p \bullet q)$
T	T				
T	F				
F	T				
F	F				

2. Determine the truth values for  $\sim(J \bullet R)$  and  $\sim J \bullet \sim R$  to prove that they are different propositions. The initial truth values of  $J$  and  $R$  should follow the same pattern as the truth values of  $p$  and  $q$  in Problem 1.

$J$	$R$	$\sim J$	$\sim R$	$(J \bullet R)$	$\sim(J \bullet R)$	$\sim J \bullet \sim R$
-----	-----	----------	----------	-----------------	---------------------	-------------------------

3. Write sentences in English (using *both* and *not*) corresponding to the two compound propositions in Problem 2, using *Joe is a student* for  $J$  and *Rachel is a student* for  $R$ .

$\sim(J \bullet R)$  \_\_\_\_\_

$\sim J \bullet \sim R$  \_\_\_\_\_

Determine the truth value for the compound propositions. Assume that propositions  $A$  and  $B$  are true, while  $X$  and  $Y$  are false. Circle T if the entire compound proposition is true. Circle F if it is false. Use the space at the right for showing any work.

4.  $\sim A \vee B$                       T    F

5.  $X \vee \sim B$                       T    F

6.  $\sim(A \vee B)$                       T    F

7.  $(A \bullet X) \vee (B \bullet Y)$             T    F

8.  $\sim[X \vee (Y \bullet \sim A)]$             T    F

*Continued on next page.*

Identify the truth value of each sentence by circling T or F. (Note that Jonah, Isaiah, and Jeremiah were all prophets.)

- |  |   |   |
|--|---|---|
| 9. Jonah was a prophet or Isaiah was a prophet.                                      | T | F |
| 10. Jeremiah was not a prophet but Isaiah was a prophet.                             | T | F |
| 11. It is not true that both Jeremiah was a prophet and<br>Isaiah was not a prophet. | T | F |
| 12. Jonah was not a prophet or both Jeremiah and Isaiah<br>were not prophets.        | T | F |
| 13. A false proposition is not true.   | T | F |
| 14. It is false that a true proposition is not false.                                | T | F |
| 15. It is true that it is false that a true proposition is not false.                | T | F |

# THE CONDITIONAL

A very useful logical operator is the **conditional** (also called **hypothetical** or **material implication**). The conditional is an *if/then*-type proposition: “If it is raining then I will take my umbrella.” The proposition following the *if* is called the **antecedent**, and the proposition following the *then* is the **consequent**. In the preceding example, “It is raining” is the antecedent; “I will take my umbrella” is the consequent.

Conditionals can take many forms. All of the following propositions can be considered as conditionals, because they can all be translated into *if/then* form:

1. If I move my rook then he will put me in check.
2. The diode will light if the switch is closed.
3. Fido is a dog implies that Fido is a mammal.
4. When you finish your dinner I will give you dessert.
5. Cheating during a test is a sufficient condition for your suspension.

Can you determine the antecedent and the consequent for each of those statements?

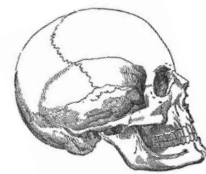
The symbol for the conditional logical operator is  $\supset$  (called a *horseshoe*). “If I move my rook then he will put me in check” could be symbolized  $R \supset C$  (read as *If R then C*).

In a conditional proposition, the antecedent is said to *imply* the consequent. That is, for a true conditional, if the antecedent is considered to be true (whether or not it actually *is* true), then the consequent must also be true. Like the disjunction, the concept of implication is somewhat ambiguous. Example one above shows that it can apply to the likelihood of behavior. The person is stating the likelihood of his opponent’s behavior when certain conditions are



## DEFINITIONS

The **conditional** operator ( $\supset$ , *if/then*) asserts that one component (the *antecedent*) implies the other (the *consequent*). It is false if and only if the antecedent is true and the consequent is false.



## CAUTION

Conditionals can take many forms other than the basic “if *x*, then *y*” structure.



### KEY POINT

The conditional includes many types of implication: cause/effect, definition, promises, conditions, and so on.

met. Some proverbs are of this type: “If a ruler pays attention to lies, all his servants become wicked” (Prov. 29:12). Example two above is a cause/effect relationship. The closing of the switch causes the diode to light. Example three is an implication by definition; all dogs are mammals. Example four refers to a promise, such as that of a parent to a child. Example five refers to a sufficient condition, in this case, the condition for a student’s suspension.

Propositional logic deals with this ambiguity by recognizing that each of the given examples are false when the antecedent is true and the consequent is false. If he moves his rook but his opponent does not put him in check, example one is false. If the switch is closed and the diode doesn’t light, example two is false, and so on. All other combinations of true and false in the conditional are considered to be true.

The defining truth table for the conditional is thus:

p	q	$p \supset q$
T	T	T
T	F	F
F	T	T
F	F	T

The last two rows may cause some problems. How can *if FALSE then TRUE* be considered true? Worse yet, how can *if FALSE then FALSE* be true? These are good questions to ask, though the answer may be hard to grasp. But consider the following examples of such propositions:



### KEY POINT

Though it may seem counterintuitive, a conditional is always true if the antecedent is false.

If a poodle is a tiger, then a poodle is a mammal.  
F T

If a poodle is a tiger, then a poodle is a feline.  
F F

Both of these conditional propositions are true. If a poodle really was a tiger (i.e., if the antecedent, though false, was *considered* to be true), then a poodle would be a mammal (which of course it is). You see that it is possible for an *if FALSE then TRUE* proposition to be true. Similarly, if a poodle really was a tiger, then it really would be a feline. This *if FALSE then FALSE* proposition is true.

Now, it is equally possible to develop *if* FALSE *then* TRUE conditionals and *if* FALSE *then* FALSE conditionals which are false. Try substituting “dog” for “mammal,” and “lizard” for “feline” in the above conditionals. This gives us the following propositions:

If a poodle is a tiger, then a poodle is a dog.  
                                   F                                  T

If a poodle is a tiger, then a poodle is a lizard.  
                                   F                                  F

Those propositions are both false. This shows that conditional propositions, as they are commonly used in everyday English, are not really truth-functional when the antecedent is false. How then are we to understand conditionals?

Another way of thinking about this is to consider  $p \supset q$  as meaning  $\sim(p \bullet \sim q)$ . So the proposition *If I move my rook then he puts me in check* is considered logically equivalent to *It is false that I move my rook and he does not put me in check*. Another example: *If you study then you will pass* is equivalent to *It is false that you study but you don't pass*. Consider these carefully and you should see how they are equivalent.

The following truth table development of  $\sim(p \bullet \sim q)$  shows that it has the same pattern as  $p \supset q$ :

p	q	$\sim q$	$(p \bullet \sim q)$	$\sim(p \bullet \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Other equivalent compound propositions could be developed which show this same T F T T pattern, as we shall see in the exercise.

As noted earlier, conditionals can take many forms. Let's look at how to best symbolize them.

First, the proposition *The diode will light if the switch is closed* places the antecedent after the consequent. This proposition means the same as *If the switch is closed then the diode will light*. In general, *p if q* means *If q then p*. However, the superficially similar proposition *p only if q* means the same as *If p then q*. This is not immediately

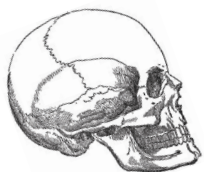
obvious, so consider this true proposition as an example: *A polygon is a square only if it has four sides.* This proposition does not mean “If a polygon has four sides, then it is a square” (which is false), but rather “If a polygon is a square then it has four sides.”

Second, *Fido is a dog implies that Fido is a mammal* is clearly just another way of saying that *If Fido is a dog then Fido is a mammal.* Similarly, *When you finish your dinner I will give you dessert* means *If you finish your dinner then I will give you dessert.* So both *p implies q* and *When p, q* are equivalent to *If p then q.*

Third, *Cheating during a test is a sufficient condition for your suspension* means that if one cheats during a test, then one will be suspended (since if one cheats during a test but is not suspended, then cheating during a test apparently is *not* sufficient). Thus, *p is sufficient for q* is equivalent to *If p then q.* On the other hand, *p is a necessary condition for q* is equivalent to *If q then p.* For example, *The presence of water is necessary for life to exist there* is best translated *If life exists there then water is present.*

Fourth, consider the proposition *p unless q.* How is this to be translated? Well, what does *You will starve unless you eat sometime* mean? A reasonable translation is “If you do not eat sometime, then you will starve.” Note, however, that this proposition is not equivalent to *If you eat sometime then you will not starve* since a person could eat something, but still starve later. So *p unless q* should be translated as *If not q then p.* However, we must be careful: When the “unless” appears at the beginning of the sentence, the translated proposition gets turned around as well. *Unless you repent, you too will perish* means “If you do not repent, then you too will perish.” So *unless p, q* is equivalent to *If not p then q.*

Finally, note that *If p then q* is equivalent to *If not q then not p.* This equivalence is called the **rule of transposition**, which we will see later. It is similar to the contrapositive of a categorical statement. So the proposition *If a whale is a mammal, then a whale breathes air* is logically equivalent to *If a whale does not breathe air, then a whale is not a mammal.*



### CAUTION

Translating conditionals can be tricky. The proposition *p only if q* means *If p then q*; *p is sufficient for q* is equivalent to *If p then q*; *p is a necessary condition for q* is equivalent to *If q then p*; and finally, *p unless q* means *If not q then p.*



### KEY POINT

*If p then q* is equivalent to *If not q then not p.*

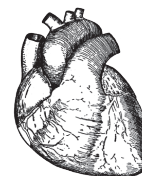
The following table summarizes the above information:

Proposition	Translation
If p then q	$p \supset q$
p implies q	$p \supset q$
p only if q	$p \supset q$
When p, q	$p \supset q$
p is sufficient for q	$p \supset q$
p if q	$q \supset p$
p is necessary for q	$q \supset p$
p unless q	$\sim q \supset p$
Unless p, q	$\sim p \supset q$
<b>Rule of transposition:</b>	$(p \supset q) \equiv (\sim q \supset \sim p)$

---

## SUMMARY

The conditional is an important logical operator. It represents *if/then* propositions and has the symbol  $\supset$ . The conditional is considered false if and only if the antecedent is true and the consequent is false. Thus,  $p \supset q$  can be considered equivalent to  $\sim(p \cdot \sim q)$ . Many different propositions can be translated as conditional propositions. They are summarized in the table above.







**EXERCISE 4** (16 points)

1. Develop the truth table for the compound proposition  $\sim p \vee q$  on the line below.

p	q	$\sim p$	$\sim p \vee q$
T	T		
T	F		
F	T		
F	F		

2. What other compound proposition has the same truth table as  $\sim p \vee q$ ?

\_\_\_\_\_

If A, B, and C represent true propositions and X, Y, and Z represent false propositions, determine whether each compound proposition is true or false, and circle the appropriate letter.

3. $A \supset B$	T	F
4. $B \supset Z$	T	F
5. $X \supset C$	T	F
6. $(A \supset B) \supset Z$	T	F
7. $X \supset (Y \supset Z)$	T	F
8. $(A \supset Y) \vee (B \supset \sim C)$	T	F
9. $[(X \supset Z) \supset C] \supset Z$	T	F
10. $[(A \bullet X) \supset Y] \supset [(X \supset \sim Z) \vee (A \supset Y)]$	T	F

If S represents *I will go swimming* and C represents *The water is cold*, symbolize each statement:

- |   |       |
|---|-------|
| 11. If the water is not cold then I will go swimming. | _____ |
| 12. I will go swimming if the water is cold.          | _____ |
| 13. I will go swimming unless the water is cold.      | _____ |
| 14. I will go swimming only if the water is not cold. | _____ |
| 15. When the water is cold I will go swimming.        | _____ |



# THE BICONDITIONAL

The final logical operator we will consider is the *biconditional*. Biconditionals represent *if and only if* propositions, such as *Skyscrapers are buildings if and only if it is false that skyscrapers are not buildings*. The symbol for biconditional is  $\equiv$ . If *Skyscrapers are buildings* is represented by the constant  $B$ , the proposition could be symbolized as  $B \equiv \sim\sim B$  (read as *B if and only if not not B*).

What does *if and only if* mean? Recall that *p only if q* means the same as *If p then q*, while *p if q* means *If q then p*. Putting these together, *p if and only if q* means *If p then q and if q then p*. The biconditional can thus be considered as the conjunction of a conditional and its converse. Taking  $p \supset q$  as the conditional and  $q \supset p$  as its converse, this means that  $p \equiv q$  is logically equivalent to  $(p \supset q) \cdot (q \supset p)$ . You can see why it is called the biconditional. We can use this equivalent proposition to develop the defining truth table for the biconditional:

p	q	$p \supset q$	$q \supset p$	$(p \supset q) \cdot (q \supset p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Thus the biconditional is true when both parts are true or when both parts are false. In other words, the biconditional is true if and only if the truth values of both parts are the same.

We can simplify the defining truth table for the biconditional like this:

p	q	$p \equiv q$
T	T	T
T	F	F
F	T	F
F	F	T

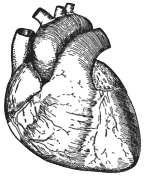


## DEFINITIONS

The biconditional operator ( $\equiv$ , “if and only if”) is true when both component propositions have the same truth value, and is false when their truth values differ.

**SUMMARY**

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The biconditional represents *if and only if* propositions, and has the symbol  $\equiv$ . The biconditional is true when both parts have the same truth value; otherwise it is false.

**EXERCISE 5** (25 points)

Given:             $A$  means *Apples are fruit*                       $C$  means *Carrots are fruit*  
                       $B$  means *Bananas are fruit*                       $D$  means *They are delicious*

Translate each symbolic proposition

1.  $A \bullet D$  \_\_\_\_\_
2.  $B \vee C$  \_\_\_\_\_
3.  $\sim C \supset \sim D$  \_\_\_\_\_
4.  $A \equiv B$  \_\_\_\_\_
5.  $(A \bullet B) \equiv \sim C$  \_\_\_\_\_  
 \_\_\_\_\_

Symbolize each compound proposition.

6. Apples and bananas are both fruit, but carrots are not fruit. \_\_\_\_\_
7. Bananas are fruit implies that they are delicious. \_\_\_\_\_
8. Carrots are fruit if and only if bananas are not fruit. \_\_\_\_\_
9. Either bananas or carrots are fruit, but they are not both fruit. \_\_\_\_\_
10. Apples are not fruit if and only if it is false that apples are fruit. \_\_\_\_\_
11. Apples are fruit is a necessary and sufficient condition for  
 bananas being fruit. \_\_\_\_\_



# LOGICAL EQUIVALENCE AND CONTRADICTION

The biconditional has another useful function beyond translating “if and only if” propositions. Since the biconditional is true whenever the truth values of the component parts are the same, the biconditional can be used to determine whether or not two propositions are **logically equivalent**; that is, it can show if two propositions have identical truth values.

Consider the example from the previous lesson, in which it was stated that  $B \equiv \sim\sim B$ . The truth table for this is:

B	$\sim B$	$\sim\sim B$	$B \equiv \sim\sim B$
T	F	T	T
F	T	F	T

This biconditional is always true, so  $B$  and  $\sim\sim B$  are seen to be logically equivalent.

A proposition that is true for every row in the truth table is called a **tautology**. In other words, tautologies are propositions that are true by logical structure. The compound proposition  $B \equiv \sim\sim B$  is thus a tautology. Other important tautologies are  $p \supset p$  and  $p \vee \sim p$ . So we can now say more briefly that the biconditional of logically equivalent propositions is a tautology.

When a proposition is false for every row in the truth table, you have a **self-contradiction**. Self-contradictions are propositions that are false by logical structure, such as  $p \bullet \sim p$ .

Consider the two propositions  $p \supset q$  and  $p \bullet \sim q$ , along with their biconditional. We will do this truth table (and every one from now on) without guide columns, simply placing the truth values immediately below the variables  $p$  and  $q$  and working out the truth value of the compound propositions, finishing with the  $\equiv$  sign.



## DEFINITION

Two propositions are **logically equivalent** if and only if they have identical truth values in a truth table.

A **tautology** is a proposition that is always true due to its logical structure.

A **self-contradiction** is a proposition that is false by logical structure.



## KEY POINT

The biconditional can be used to test for equivalence. If the biconditional of two statements is a tautology, then the statements are equivalent.



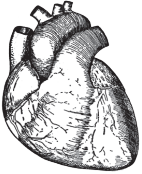
### KEY POINT

The biconditional can be used to test for contradiction. If the biconditional of two statements is a self-contradiction, then the statements are contradictory.

$$\frac{(p \supset q) \equiv (p \bullet \sim q)}{\begin{array}{cccc} T & T & T & F \\ T & F & F & F \\ F & T & T & F \\ F & T & F & F \end{array}}$$

Because this biconditional is a self-contradiction, we can say that  $p \supset q$  contradicts  $p \bullet \sim q$ .

### SUMMARY



When the biconditional of two propositions is a tautology, the propositions are logically equivalent. When it is a self-contradiction, the propositions are contradictory. A tautology is a proposition that is true by logical structure. A self-contradiction is a proposition that is false by logical structure.



**EXERCISE 6** (30 points)

Set up the biconditional between each pair of propositions (as in the lesson) to determine if they are *logically equivalent*, *contradictory*, or *neither*. In this exercise, do not use guide columns. Rather, place the truth values immediately beneath the variables and work through the proposition to determine its truth value. Problem 4 has three variables, so it will require eight rows.

1.  $\sim (p \vee q) \quad \sim p \vee \sim q$

2.  $p \supset q \quad \sim q \supset \sim p$

3.  $\sim (\sim p \vee q) \quad p \supset q$

4.  $p \supset (q \supset r) \quad (p \supset q) \supset r$

5. Write a set of propositions in English which could be represented by the symbolic propositions in Problem 2.

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# TRUTH TABLES FOR DETERMINING VALIDITY

So far we have seen three uses for truth tables: determining the truth values of compound propositions, defining logical operators, and determining logical equivalence. Another use for truth tables, and perhaps the most practical (and thus the most interesting), is that of determining the validity of propositional arguments.

Before we look at how truth tables do this, we first need to consider what is meant by **validity**. When an argument is valid, the conclusion follows necessarily from the premises. In other words, *if the premises are assumed to be true, then in a valid argument the conclusion must also be true*. If an argument has true premises with a false conclusion, it is invalid.

To use truth tables to determine the validity of an argument, the argument is translated into symbolic form (if it is not already symbolic) then placed above a line, with the symbol  $\therefore$  (meaning “therefore”) in front of the conclusion. Then the truth values are placed below the propositions, just like we have done before. These steps are completed for the *modus ponens* argument shown:

$p \supset q$	$p$	$\therefore q$
T	T	T
F	T	F
T	F	T
T	F	F

Notice that  $p$  and  $q$  (and thus  $p \supset q$ ) have the same pattern of T and F that we have seen up to this point.

Now consider again the definition of validity. How does the truth table show that this argument is valid? Well, the argument must be either valid or invalid. If it was invalid, there would be a horizontal



## DEFINITIONS

Recall that in a **valid** argument, if the premises are true, the conclusion must be true. If the premises can be true and the conclusion false, the argument is **invalid**.



## KEY POINT

The symbol  $\therefore$  means “therefore” and signals the conclusion of an argument.



### KEY POINT

If a truth table shows at least one row in which the premises of an argument are true but the conclusion is false, then the argument is invalid. Otherwise, it is valid.

row which showed *true premises* with a *false conclusion*. No such row exists; the argument is not invalid. So it must be valid.

Consider this another way. Look at each of the four rows for the above argument. For *every* row in which all the premises show T, does the conclusion also show T? Yes it does; the first row above is the only row with all true premises, and it also shows a true conclusion. The argument is thus valid. If *any* row showed premises with all Ts and a conclusion with F, it would be invalid, even if *other* rows had premises with all Ts and a conclusion with a T.

For an example of an argument shown to be invalid by truth table, consider the *denying the antecedent* argument here:

$p \supset q$	$\sim p$	$\therefore \sim q$
T	F	F
F	F	T
T	T	F
T	T	T

← INVALID

The truth values have been completed for the premises and conclusion (with the initial truth values for  $p$  and  $q$  removed for the sake of clarity). Now, notice that the third row has true premises with a false conclusion. This argument is thus invalid (even though the fourth row shows true premises with a true conclusion). To mark it as invalid, identify the row (or rows) with true premises and a false conclusion and write *Invalid* near it, as shown above.

Let's look at two more examples of truth tables for validity, one valid and one invalid. On the first one we will show the step-by-step procedure. Consider, just for fun, the argument above: "The argument must be either valid or invalid. If it is invalid then there will be a row of true premises with a false conclusion. There is no row of true premises with a false conclusion. Therefore the argument is valid." We symbolize this argument as  $V \vee I, I \supset R, \sim R, \therefore V$ , and complete the truth table. First, write out the argument in symbolic form, placing the truth values under the three constants  $V, I$ , and  $R$  in the same pattern as we used before:

# UNIT ONE: TRUTH TABLES

$V \vee I$	$I \supset R$	$\sim R$	$\therefore V$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

From these truth values, determine the other values until the entire argument is completed. You may find it easier to start from the right and work your way left.

$V \vee I$	$I \supset R$	$\sim R$	$\therefore V$
T T T	T T T	FT	T
T T T	T F F	TF	T
T T F	F T T	FT	T
T T F	F T F	TF	T
F T T	T T T	FT	F
F T T	T F F	TF	F
F F F	F T T	FT	F
F F F	F T F	TF	F

Then the unnecessary columns of T and F may be removed (by erasing or marking out), leaving only the patterns for the premises and the conclusion, as shown:

$V \vee I$	$I \supset R$	$\sim R$	$\therefore V$
T	T	F	T
T	F	T	T
T	T	F	T
T	T	T	T ← VALID
T	T	F	F
T	F	T	F
F	T	F	F
F	T	T	F

The only row with all true premises is the fourth row down, and it also shows a true conclusion. Thus the argument is valid, as marked.

Now for one more example of an invalid argument before we go on to the assignment. “If we stop here then I will be lost. If we stop

here then you will be lost. So either I will be lost, or you will.” This can be symbolized  $S \supset I, S \supset Y, \therefore I \vee Y$ .

The truth table can be developed as before (you should do so on your own), resulting in the following patterns:

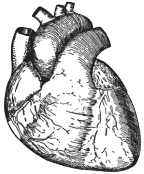
$S \supset I$	$S \supset Y$	$\therefore I \vee Y$
T	T	T
T	F	T
F	T	T
F	F	F
T	T	T
T	T	T
T	T	T
T	T	F

← INVALID

Although there are many rows which have true premises with a true conclusion (namely rows one, five, six and seven), the eighth row shows true premises with a false conclusion. Thus the entire argument has been shown to be invalid, and is marked as such.

We have the following procedure for determining the validity of arguments using truth tables.

## SUMMARY



### The Truth Table Method for Validity

1. Write the argument in symbolic form on a line.
2. Under the variables, place the columns of T and F.
3. Determine the columns of T and F for the propositions following the defining truth tables.
4. Remove any unnecessary columns of T and F, leaving only the columns for the premises and conclusion.
5. Examine the rows. If any row has all true premises with a false conclusion, the argument is invalid. Otherwise it is valid. Mark the row(s) showing valid or invalid.

**EXERCISE 7a** (48 points)

Determine the truth value of each compound proposition. Assume that propositions A and B are true, X and Y are false, and P and Q are unknown. Circle T if the proposition is true, F if it is false, and ? if the truth value cannot be determined. (Hint: There are two of each.)

- |  |   |   |   |
|--|---|---|---|
| 1. $P \vee \sim P$   | T | F | ? |
| 2. $(P \supset P) \supset \sim A$                                | T | F | ? |
| 3. $(Y \supset P) \supset Q$                                     | T | F | ? |
| 4. $P \equiv (X \vee Y)$   | T | F | ? |
| 5. $\sim Q \cdot [(P \vee Q) \cdot \sim P]$                      | T | F | ? |
| 6. $\sim[P \vee (B \cdot Y)] \vee [(P \vee B) \cdot (P \vee Y)]$ | T | F | ? |

Use truth tables to determine the validity of each argument. Identify each as either VALID or INVALID, and identify the rows that show this.

7.  $\underline{p \supset q} \quad \therefore \underline{p \supset (p \cdot q)}$

8.  $\underline{p \cdot q} \quad \therefore \underline{p \vee q}$

9.  $\underline{p \equiv \sim q} \quad \underline{\sim q} \quad \therefore \underline{p}$

10.  $\underline{p \vee q} \quad \underline{\sim p} \quad \therefore \underline{q}$

*Continued on next page.*

## INTERMEDIATE LOGIC

Translate the arguments into symbolic form using the given constants, and then use truth tables to determine their validity as in the previous problems.

11. If Jesus was John the Baptist raised from the dead, then He could do miracles. Jesus did miracles, so He was John the Baptist raised from the dead. (J means *Jesus was John the Baptist raised from the dead*, M means *He could do miracles*.)
12. If Jeff studies then he will get good grades. If Jeff does not study then he will play. So Jeff will either get good grades or he will play. (S means *Jeff studies*, G means *He will get good grades*, P means *He will play*.)
13. If Jesus is not God then He was a liar or He was insane. Jesus was clearly not a liar. He certainly was not insane. We conclude that Jesus is God. (G means *Jesus is God*, L means *He was a liar*, I means *He was insane*.)

*Continued on next page.*



UNIT ONE: TRUTH TABLES

14. If taxes increase then the public will complain, but if the deficit increases then the public will complain. Either taxes or the deficit will increase. Thus the public is bound to complain. (T means *Taxes increase*, P means *The public will complain*, D means *The deficit increases*.)



**EXERCISE 7b** (30 points)

Use truth tables to determine the validity of the propositional arguments below. (Problems 4 and 5 require eight rows each; Problem 6 requires sixteen!)

1.  $p \quad \therefore \sim p \vee q$

2.  $p \supset q \quad \therefore \sim q \supset \sim p$

3.  $p \supset q \quad \sim q \quad \therefore p \equiv q$

4.  $p \supset (q \supset r) \quad q \quad \therefore r \supset p$

5.  $p \supset (\sim q \supset r) \quad p \quad \therefore \sim r \supset q$

$$6. \quad \frac{(p \supset q) \cdot [(p \cdot q) \supset r] \quad p \supset (r \supset s)}{\therefore p \supset s}$$

# SHORTER TRUTH TABLES FOR DETERMINING VALIDITY

In the last exercise, you found that you needed to write hundreds of Ts and Fs because of the number of variables. And each time a new variable is added, the size of the truth table doubles. With this level of complexity it is easy to get confused or make careless errors. Surely there must be a shorter method!

Fortunately, there is. All the work in a truth table can (for most arguments) be compressed into only one row. That's right, just one. In this lesson we will see how it works.

Remember that an argument is proved invalid whenever the premises can be shown to be true and the conclusion false. With the shorter truth table, you start by *assuming the argument to be invalid*. You assume each premise is true and the conclusion is false. Then, you work backwards along the argument, trying to make this assumption work without any contradictions. If you succeed, you have proved the argument to be invalid. However, if assuming the argument to be invalid results in an unavoidable contradiction, then your assumption is wrong and the argument must be valid.

Take, for example, one of the arguments from the last chapter. We start by assigning the premises the value T and the conclusion the value F. Notice that the Ts and F are placed under those parts of the propositions which would be filled in last in the longer truth table.

$S \supset I$	$S \supset Y$	$\therefore I \vee Y$
T	T	F

Now, for the disjunction  $I \vee Y$  to be false as assumed, both disjuncts must be false, according to the defining truth table. But if I and Y are false in the conclusion, they must be false in the premises. Thus we obtain this table:



## KEY POINT

The validity of most arguments can be determined with a truth table having only one row. Assume the conclusion is false and the premises true, then work backward looking for unavoidable contradictions.

$S \supset I$	$S \supset Y$	$\therefore I \vee Y$
T F	T F	F F F

A true conditional with a false consequent must also have a false antecedent (check the defining truth table for the conditional). Thus we assign the antecedents  $S$  in the above conditionals the value of F, as shown:

$S \supset I$	$S \supset Y$	$\therefore I \vee Y$	
F T F	F T F	F F F	INVALID

We are now finished. We assumed the argument was invalid, every truth value was determined, and no contradiction was found. Thus we conclude the argument is indeed invalid.

Now we will look at the valid argument from the last chapter. Again, we start by assuming the argument to be invalid (true premises, false conclusion), then work backward to see if we get a contradiction.

$V \vee I$	$I \supset R$	$\sim R$	$\therefore V$
T	T	T	F

If  $V$  is false in the conclusion, it must be false everywhere else. Write F under the  $V$  in the first premise. Also, if  $\sim R$  is true, then  $R$  must be false. We write F under the  $R$ s and get

$V \vee I$	$I \supset R$	$\sim R$	$\therefore V$
F T	T F	T F	F

Now, look at  $I \supset R$ . For this conditional to be true with a false consequent, the antecedent  $I$  must be false. And if  $I$  is false there, then it is false in  $V \vee I$ . Filling these in gives us

$V \vee I$	$I \supset R$	$\sim R$	$\therefore V$	
F T F	F T F	T F	F	VALID
$\uparrow$ CONTRADICTION				

We see that  $V$  and  $I$  are both found to be false. But this would imply the disjunction  $V \vee I$  is false. However, we assigned it as a premise the value of true. This contradiction means that it is impossible to make the argument invalid. Thus it must be valid.

Now for two familiar examples. First, consider the *modus tollens* argument  $p \supset q, \sim q, \therefore \sim p$ . We will assume it to be invalid, like this:

$p \supset q$	$\sim q$	$\therefore \sim p$
T	T	F

Start with the conclusion. If  $\sim p$  is false as assumed, then  $p$  is true. Filling that in gives us

$p \supset q$	$\sim q$	$\therefore \sim p$
T T	T	FT

But if  $\sim q$  is true,  $q$  must be false:

$p \supset q$	$\sim q$	$\therefore \sim p$	
T T F	TF	FT	VALID
↑	CONTRADICTION		

We see the contradiction in the first premise, mark it as a contradiction and write “VALID.”

For a final example, let's look at *affirming the consequent*. We assume it to be invalid:

$p \supset q$	$q$	$\therefore p$
T	T	F

We see that  $p$  is false and  $q$  is true, and write those values in.

$p \supset q$	$q$	$\therefore p$	
F T T	T	F	INVALID

There is no contradiction. The argument is invalid, with true premises and a false conclusion.

When a shorter truth table is completed for an invalid argument as above, you should discover that the truth values found for the variables (or constants) are the same truth values from one of the rows showing the argument invalid on the longer truth table. In this case, the argument was seen to be invalid when  $p$  is false and  $q$  is true. Compare this with the longer truth table:



### KEY POINT

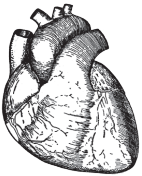
When a shorter truth table is completed for an invalid argument, the truth values found for the variables (or constants) are the same truth values from a row showing the argument to be invalid on the longer truth table.

$p \supset q$	$q$	$\therefore p$	
T	T	T	
F	F	T	
T	T	F	← INVALID
T	F	F	

We see that the longer truth table also shows the argument to be invalid when  $p$  is false and  $q$  is true.

Thus we have the following procedure for determining the validity of arguments using the shorter truth table:

## SUMMARY



### The Shorter Truth Table Method for Validity

1. Write the argument in symbolic form on a line.
2. Assume the argument is invalid by assigning the premises the value T and the conclusion the value F.
3. Work backwards along the argument, determining the remaining truth values to be T or F as necessary, avoiding contradiction if possible.
4. If the truth values are completed without contradiction, then the argument is invalid as assumed.
5. If a contradiction is unavoidable, then the original assumption was wrong and the argument is valid.



**EXERCISE 8** (48 points)

Determine the validity of each argument using the shorter truth-table method. Use the constants given in order of appearance in the argument to symbolize each proposition.

1. If I study for my test tonight then I am sure to pass it, but if I watch TV then I will get to see my favorite show. So if I study for the test and watch TV, then I will either pass the test or I will see my favorite show. (S, P, W, F)
  
2. If Caesar had been a benevolent king, then all Romans would have received their full rights under the law. The Roman Christians were persecuted for their faith. If all Romans had received their full rights, then the Roman Christians would not have been persecuted. Therefore Caesar was not a benevolent king. (B, R, P)
  
3. If a composition has both meter and rhyme, then it is a poem. It is not the case that this composition has meter or rhyme. Therefore this composition is not a poem. (M, R, P)
  
4. If the book of Hebrews is Scripture then it was written by Paul or Apollos. If Paul wrote anonymously to the Hebrews then he wrote anonymously in some of his letters. If Hebrews was written by Paul then he wrote anonymously to the Hebrews. Paul did not write anonymously in any of his letters. The book of Hebrews is Scripture. Therefore Hebrews was written by Apollos. (S, P, A, H, L)

*Continued on next page.*

5. If you sin apart from the law then you will perish apart from the law, but if you sin under the law then you will be judged by the law. If you sin, then you either sin apart from the law or you sin under the law. You do sin. Therefore you will either perish apart from the law or you will be judged by the law. (A, P, U, J, S)
  
6. If you obey the law then you will not be condemned. You have not obeyed the law. Thus, you will be condemned. (O, C)

Determine the validity of each argument from Exercise 7b using the shorter truth-table method.

7.  $\underline{p \therefore \sim p \vee q}$

8.  $\underline{p \supset q \therefore \sim q \supset \sim p}$

9.  $\underline{p \supset q \quad \sim q \therefore p \equiv q}$

10.  $\underline{p \supset (q \supset r) \quad q \therefore r \supset p}$

11.  $\underline{p \supset (\sim q \supset r) \quad p \therefore \sim r \supset q}$

12.  $\underline{(p \supset q) \cdot [(p \cdot q) \supset r] \quad p \supset (r \supset s) \therefore p \supset s}$

# USING ASSUMED TRUTH VALUES IN SHORTER TRUTH TABLES

The propositional arguments which have been examined so far have avoided one difficulty which may arise while using the shorter truth-table method. To understand what that difficulty is, we will analyze the following argument: “It is false that both reading and skiing are dangerous activities. Therefore neither reading nor skiing is dangerous.” This argument follows the form  $\sim(p \bullet q), \therefore \sim(p \vee q)$ . If we begin using the shorter truth table to determine validity we get to this point:

$\sim(p \bullet q)$	$\therefore \sim(p \vee q)$
T F	F T

Now we are stuck. For the conjunction to be false, either  $p$  or  $q$  could be false, and for the disjunction to be true, either  $p$  or  $q$  could be true. For this situation, in which there are no “forced” truth values, we must *assume a truth value*. In other words, we need to guess. Looking at the conclusion, we will guess that  $p$  is true. Working this out leads us to this:

$\sim(p \bullet q)$	$\therefore \sim(p \vee q)$	
T T F F	F T T F	INVALID
	↑ GUESS	

Our guess allowed us to find a way to make the premises true and the conclusion false, and thus determine that the argument is invalid, without having to go any further. In fact, any guess we could have made would have worked with this example. Try another guess before you go on.

Let’s look at a different example. Consider this argument:

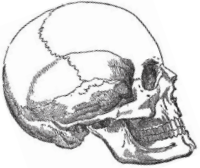


## KEY POINT

In some cases, shorter truth tables may not be completed in only one row—you might be faced with a choice in assigning truth values. In this case, you must guess at truth values until you obtain the desired result.

$p \equiv q$	$q \supset r$	$\therefore p \equiv r$
T	T	F

After taking the first step we are already stuck. There are two ways the biconditional can be true, two ways it can be false, and three ways for the conditional to be true. So we must guess. Like before, we will guess that  $p$  is true. Following the procedure leads us to obtain this:



## CAUTION

If you reach a contradiction after an initial guess, you still must try other possible guesses before concluding that an argument is valid.

$p \equiv q$	$q \supset r$	$\therefore p \equiv r$
T T F	F T F	T F F
$\uparrow$ CONTRADICTION		$\uparrow$ GUESS

We get a contradiction in the first premise, which appears to imply that the argument is valid. However, it may simply mean that we made a bad assumption. Whenever a contradiction is reached after the first guess, we must then try the other way. So we will now assume that  $p$  is false, which leads to us this:

$p \equiv q$	$q \supset r$	$\therefore p \equiv r$	
F T F	F T T	F F T	INVALID
		$\uparrow$ GUESS	

This second guess gave us no contradiction. This means that, in fact, the argument *is* invalid. You see the importance of guessing both truth values for the variable if a contradiction is found the first time.

Since the examples in this section have been found invalid, you may get the mistaken notion that any time you have to guess a truth value, the argument is necessarily invalid. This is not true. Consider this argument:  $p \supset q, q \supset p, \therefore p \equiv q$ . The shorter truth table requires one to guess a truth value. Let's start by guessing that  $p$  is true. This leads to the following result:

$p \supset q$	$q \supset p$	$\therefore p \equiv q$
TT F	F TT	T FF
$\uparrow$ CONTRADICTION		$\uparrow$ GUESS

The guess of  $p$  as true lead us to a contradiction. So we must try guessing that  $p$  is false. That leads to this result:

## UNIT ONE: TRUTH TABLES

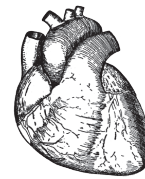
$p \supset q$	$q \supset p$	$\therefore p \equiv q$
F T T	T T F	F F T
CONTRADICTION	↑	↑ GUESS

After some experience with this method, you should find that your guesses become less random and more educated, and that you are able to determine invalid arguments to *be* invalid after the first guess. This may take some careful thought and practice, so don't get discouraged on the way.

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### SUMMARY

Sometimes when using the shorter truth-table method for validity, no forced truth values occur before you finish. When this happens, you must guess the truth value of one variable or constant, then continue with the same method. If no contradiction appears, the argument is invalid. If a contradiction does appear, you must guess the other truth value for that variable or constant.





**EXERCISE 9** (20 points)

Use the shorter truth table method to determine the validity of the following arguments. Most of these (but not all) will require you to guess a truth value.

1.  $\underline{p \equiv q \quad q \equiv r \quad \therefore p \equiv r}$

2.  $\underline{p \vee q \quad \therefore p \bullet q}$

3.  $\underline{p \supset q \quad q \equiv r \quad \therefore p \supset r}$

4.  $\underline{(p \supset q) \vee (r \supset s) \quad p \vee r \quad \therefore q \vee s}$

5.  $\underline{p \vee q \quad \sim [q \bullet (r \supset p)] \quad \therefore \sim (p \equiv q)}$

6.  $\underline{p \supset (q \supset r) \quad q \supset (p \supset r) \quad \therefore (p \vee q) \supset r}$





# SHORTER TRUTH TABLES FOR CONSISTENCY

We have seen that the shorter truth table is a powerful tool for quickly determining the validity of even relatively complex arguments. Shorter truth tables may also be used to determine the consistency of sets of propositions and the equivalence of two propositions. Let's look at consistency first.

To say that propositions are **consistent** simply means that they can be true at the same time. Assuming consistent propositions all to be true will result in no logical contradiction.

For example, consider these two propositions. "It is false that increasing inflation implies a thriving economy"; "If inflation is not increasing then the economy is not thriving." Are these propositions consistent? Can they both be true at the same time? How can we use the shorter truth table to find out? Try to answer these questions before you read on.

These two propositions can be abbreviated this way:  $\sim(I \supset E)$ ,  $\sim I \supset \sim E$ . If they are consistent, then assuming that they are both true should result in no contradiction. So let's do that. As before, the propositions are symbolized and placed above a line. Then below each proposition place a T, implying that both propositions are true, like this:

$\sim(I \supset E)$	$\sim I \supset \sim E$
T	T

Now, will this assumption run us into a contradiction? To find out, we determine the forced truth values. Since the second proposition is a conditional which can be true for three out of four combinations of true and false, we can't really do anything with it. But if the first proposition is true, then the conditional  $I \supset E$  must be false. This would imply that I is true and E is false. Carry these truth values



## KEY POINT

If you get a contradiction when assuming that two or more propositions are all true, then they are not consistent.

over to the other proposition and continue this procedure, and you should end up with this:

$\sim(I \supset E)$	$\sim I \supset \sim E$
T T F F	F T T T F

Assuming that the propositions were all true resulted in no contradiction. Thus they are consistent; they can all be true at the same time.

Now, suppose an attorney at first declared, “My client did not take those papers. The secretary took them.” Then later he admitted, “It is false that the secretary took the papers if my client did not.” Can his statements all be true? Let’s find out. These propositions can be symbolized as follows:

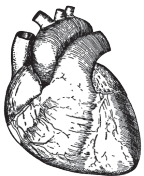
$\sim C$	$S$	$\sim(\sim C \supset S)$
T	T	T

We assume the attorney’s propositions are consistent. Does this lead us to a contradiction? Follow the shorter truth table procedure, and you should end up here:

$\sim C$	$S$	$\sim(\sim C \supset S)$
T	T	T T T T
		↑ CONTRADICTION
		INCONSISTENT

If  $\sim C \supset S$  is true, then the third proposition must be false. Thus they cannot all be true; the propositions are **inconsistent**.

## SUMMARY



Propositions are consistent when assuming them all to be true involves no contradiction. Thus the shorter truth table can be used to determine consistency by making this assumption and checking for a contradiction.

**EXERCISE 10** (25 points)

Problems 1–5: Use the shorter truth table method to determine the consistency of each proposition set.

1.  $\underline{p \quad \sim p \supset r}$

2.  $\underline{\sim \sim p \quad \sim p \cdot q}$

3.  $\underline{p \supset q \quad p \quad \sim q}$

4.  $\underline{p \vee q \quad \sim p}$

5.  $\underline{p \equiv q \quad q \equiv r \quad p \quad \sim r}$

Symbolize the propositions using the constants given, and then determine their consistency.

6. Mr. Copia owns a Porsche and a mansion. If he owns a mansion then either he owns a Porsche or I am imagining things. I am not imagining things. (P, M, I)

7. If I can use rhetoric then I learned grammar and logic. I did not learn logic but I can use rhetoric. (R, G, L)



# SHORTER TRUTH TABLES FOR EQUIVALENCE

The shorter truth table for equivalence works in a similar way as the shorter truth table for validity. In this method, we assume the two propositions are not logically equivalent, then check to see if that assumption runs us into a contradiction or not. If it does not, then our assumption is correct and they are not equivalent. However, if assuming they are not equivalent always results in a contradiction, then they must be equivalent.

Consider these two propositions: “If salt is dissolved in water then if an egg is placed in the salty water then it will float.” “If salt is dissolved in water and an egg is placed in it, then the egg will float.” Are they equivalent?

We symbolize the propositions and place them on a line. Then we assume they are not equivalent. How? By assuming one is true and the other false, as such:

$$\frac{S \supset (E \supset F) \quad (S \bullet E) \supset F}{T \qquad F}$$

Now determine the forced truth values and check for a contradiction. Doing so results in

$$\frac{S \supset (E \supset F) \quad (S \bullet E) \supset F}{TT \quad TF \quad F \quad TTT \quad FF}$$

↑ CONTRADICTION

The contradiction seems to imply that our assumption of non-equivalence was wrong. However, we also need to check the other combination of true and false for non-equivalence. That is, we now should assume the first proposition is false and the second is true. Such an assumption leads us to this point:



## KEY POINT

To test equivalence using shorter truth tables, assume the two propositions are not logically equivalent, then check to see if that leads to an unavoidable contradiction.

$$\begin{array}{c}
 \frac{S \supset (E \supset F) \quad (S \bullet E) \supset F}{\begin{array}{cc} T F & T F F \end{array} \quad \begin{array}{cc} T T T & T F \end{array}} \\
 \uparrow \text{CONTRADICTION}
 \end{array}$$

We tried both possibilities for the propositions to not be equivalent: the first proposition true and the second false, and vice versa. Both attempts wound up in a contradiction, so the assumption was wrong and the propositions are equivalent.

For another example, consider these propositions: “If the lock is broken then the door won’t open.” “The lock is not broken and the door opens.” To determine their equivalence we symbolize them and assume one to be true and the other false. Try to figure out which you should assume true and which false first.

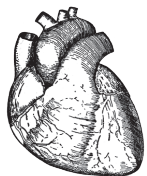
If we first assume that the conditional is false and the conjunction is true, we end up with a contradiction (try it!). However, if we assume the conditional is true and the conjunction false, we can get to this point:

$$\begin{array}{c}
 \frac{L \supset \sim O \quad \sim L \bullet O}{\begin{array}{cc} F T & T F \end{array} \quad \begin{array}{ccc} T & F & F \end{array}}
 \end{array}$$

The truth values are all assigned and there are no contradictions. The conditional is true and the conjunction is false, thus they are not equivalent.

The following procedure summarizes our method for testing the equivalence of two propositions using shorter truth tables.

## SUMMARY



### The Shorter Truth Table Method for Equivalence:

1. Write the two propositions in symbolic form on a line.
2. Assume the propositions are not equivalent by assigning one to be T and the other F.
3. If no contradiction occurs, the propositions are not equivalent.
4. If a contradiction is unavoidable, then switch the assigned truth values and try again.
5. If a contradiction is still unavoidable, then they are equivalent. However, if it is possible to avoid a contradiction, the propositions are not equivalent.

**EXERCISE 11** (20 points)

Use the shorter truth table method to determine the equivalence of each pair of propositions.

1.  $\underline{\sim(p \cdot q) \quad \sim p \vee \sim q}$

2.  $\underline{p \supset q \quad p \supset (p \cdot q)}$

3.  $\underline{p \vee (p \supset q) \quad q \supset p}$

4.  $\underline{p \quad p \vee (p \cdot q)}$

5. If Christ's righteousness is not imputed to you, then you are condemned. Christ's righteousness is imputed to you or you are condemned.





# THE DILEMMA

Any argument presenting two alternatives, either of which when chosen leads to certain conclusions, may be called a **dilemma**. The dilemma is often used to trap an opponent in debate. It is also a common way of thinking when we are trying to decide what course to take between two apparently opposing options.

For example, you might find yourself reasoning like this: “If I go to college then I delay making money, but if I go straight into business then I will get a low-paying job. I will either go to college or straight into business, so I will either delay making money or I will get a low-paying job.” This argument follows the general form of a **constructive dilemma**:

$$(p \supset q) \cdot (r \supset s) \quad p \vee r \quad \therefore q \vee s$$

A similar type of argument is the **destructive dilemma**, which follow this form:

$$(p \supset q) \cdot (r \supset s) \quad \sim q \vee \sim s \quad \therefore \sim p \vee \sim r$$

Here is an example of such a destructive dilemma: “If something can be done, then it is possible, and if it can be done easily, then it is likely. Faster-than-light travel is either impossible or unlikely, so it either cannot be done, or it cannot be done easily.”

You can see that constructive dilemmas are sort of an extended *modus ponens*, while destructive dilemmas are like *modus tollens*.

Let’s look at a few specific dilemma types. In one constructive type, the antecedent of one conditional is the negation of the antecedent of the other:

$$(p \supset q) \cdot (\sim p \supset r) \quad p \vee \sim p \quad \therefore q \vee r$$



## DEFINITIONS

A **dilemma** is a valid argument which presents a choice between two conditionals.



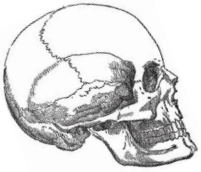
## KEY POINT

Constructive dilemmas work like *modus ponens*, while destructive dilemmas work like *modus tollens*.

Because the second premise is a tautology, it is often left unstated. Here is such an argument, the premises of which come from Proverbs 26:4–5: “If you answer a fool according to his folly, then you will be like him. However, if you do not answer a fool according to his folly, then he will be wise in his own eyes. Therefore no matter how you answer a fool, you will either be like him or he will be wise in his own eyes.”

In another type of constructive dilemma, the consequent of each conditional is the same, resulting in the following argument:

$$(p \supset q) \bullet (r \supset q) \quad p \vee r \quad \therefore q \vee q$$



### CAUTION

Dilemmas can appear in many special forms—make sure you understand the differences between them.

In this case the conclusion  $q \vee q$  is equivalent to  $q$ , and is usually stated that way. For example: “If Congressman Jones lied about the sale of arms then he should not be re-elected. Neither should he be re-elected if he honestly couldn’t remember something so important. He either lied or he couldn’t remember, so he should not be re-elected.”

Consider this dilemma: “If this bill is to become a law then it must pass the Congress and the President must sign it. But either it will not make it through Congress or the President will not sign it. Therefore this bill will not become a law.” In symbolic form this follows the pattern

$$p \supset (q \bullet r) \quad \sim q \vee \sim r \quad \therefore \sim p$$

You should be able to show that this is equivalent to a destructive dilemma which has the same antecedent  $p$  for both conditionals.

The ability to produce a good dilemma is useful in debate, as is the ability to get out of a dilemma being used against you. Using the shorter truth table, we can easily prove these various dilemmas to be valid. How can we avoid the conclusion of a valid argument? One way is to claim that, though valid, the argument is not sound; that is, one or both of the premises is false. Another way is to produce a similar argument that may be used to prove something else.

Facing a dilemma has been picturesquely referred to as being “impaled on the horns of a dilemma,” as if it were a charging bull. Three main options are usually presented for escaping the horns of a dilemma:

1. You could **go between the horns**, meaning you could deny the disjunctive premise and provide a third alternative, somewhere in

the middle. In the first example, someone could reply, “The choice isn’t between college or a full-time business. You could go to college part time and work part time.” The disjunction is charged with being an *either/or* fallacy (i.e., a false dilemma).

2. You could **grasp it by the horns**. This is done by rejecting one of the conditionals in the conjunctive premise. For example, with the dilemma about the bill above you could reply, “Even if the president refuses to sign it, the Congress could still override his veto with a two-thirds majority.” And if one conjunct is false then the entire conjunction is false.

3. Finally, you could **rebut the horns** with a counter-dilemma. A counter-dilemma which is made up of the same components as the original dilemma is usually the most rhetorically effective. Consider the dilemma about answering a fool. One possible counter-dilemma is, “If you answer a fool according to his folly, then he will not be wise in his own eyes. And if you do not answer a fool according to his folly, then you will not be like him. Therefore he will either not be wise in his own eyes or you will not be like him.” Notice that the counter-dilemma does not claim that the original dilemma is false or invalid. It simply is another way of looking at the facts in order to arrive at a different conclusion.

Let’s consider one more example, and see how all three of these methods could be used against it. Suppose your friend complained, “If I study for the test then I’ll miss my favorite show. But if I don’t study then I’ll fail the test. I will either study or not study, so I’ll either miss my favorite show or I’ll fail the test.” How could you answer him?

First, you could go between the horns by saying, “You could study for the test a little while before your show comes on, then study a little before class tomorrow.” Second, you could grasp his dilemma by the horns, saying “If you don’t study you won’t necessarily fail, not if you have been paying attention in class.” Third, you could confront him with this counter-dilemma: “If you study for the test then you will surely pass, and if you don’t study then you’ll get to see your favorite show. Either you will study or not, so you will either pass the test or you will get to see your favorite show!”

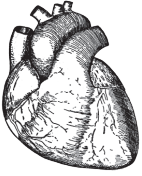


### KEY POINT

There are three main ways to escape the horns of a dilemma: go between the horns, grasp the horns, or rebut the horns.

**SUMMARY**

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A dilemma is an argument presenting a choice between two conditionals joined by conjunction. The two main types of the dilemma are constructive and destructive. There are three means of avoiding being impaled on the horns of a dilemma: go between the horns by denying the disjunction, grasp it by the horns by denying a conditional, or rebut the horns by means of a counter-dilemma.

**EXERCISE 12** (18 points)

Symbolize the dilemma from the end of this lesson about studying or watching your favorite show. Then symbolize the counter-dilemma below it. In the space below the lines, use shorter truth tables to demonstrate the validity of both arguments.

1. The dilemma: \_\_\_\_\_

2. The counter-dilemma: \_\_\_\_\_

Refute each of the following dilemmas using the given methods.

3. If angels are material, then they cannot all simultaneously fit on the head of a pin. If angels are immaterial, then they can neither dance nor be in contact with the head of a pin. Angels are either material or immaterial. Either way, all the angels that exist cannot simultaneously dance on the head of a pin. (Grasp the horns.)

\_\_\_\_\_  
\_\_\_\_\_

4. If Congressman Jones lied about the sale of arms then he should not be re-elected. Neither should he be re-elected if he honestly couldn't remember something so important. He either lied or he couldn't remember, so he should not be re-elected. (Go between the horns.)

\_\_\_\_\_  
\_\_\_\_\_

5. If you sin apart from the law then you will perish apart from the law, but if you sin under the law then you will be judged by the law. You either sin apart from the law or you sin under the law. Therefore you will either perish apart from the law or you will be judged by the law. (Grasp the horns.)

\_\_\_\_\_  
\_\_\_\_\_

*Continued on next page.*

6. If God were perfectly good then He would always be willing to prevent evil, and if God were infinitely powerful then He would always be able to prevent evil. But God is either unwilling or unable to prevent evil. Therefore He is either not perfectly good or He is not infinitely powerful. (Grasp the horns.)

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7. If teachers cover more material, the students will be more confused. If teachers cover less material, the students will not learn as much. Teachers will cover more or less material, so either students will be confused or they will not learn as much. (Rebut the horns.)

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8. If taxes increase then the public will complain, but if the deficit increases then the public will complain. Either taxes or the deficit will increase. Thus the public is bound to complain. (Use any or all three methods.)

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# REVIEW QUESTIONS

*Answers can be found in the lesson under which the questions are listed.*

## **Introduction**

How is logic a science? How is it an art? How is it a symbolic language? What becomes more apparent about an argument when it is symbolized? How does propositional logic differ from categorical logic?

## **Lesson 1: Introduction to Propositional Logic**

What is a proposition? What is a truth-functional proposition? Why is a self-report not truth-functional? What is a logical operator? How does a simple proposition differ from a compound proposition? What is a propositional constant? What is a propositional variable?

## **Lesson 2: Negation, Conjunction, and Disjunction**

How is negation expressed in a sentence in regular English? What is the symbol for negation? How does negation affect the truth value of the negated proposition? What is a defining truth table? What English words express a conjunction? What is the symbol for conjunction? When is a conjunction true? How is a disjunction expressed in regular English? What is the difference between “inclusive or” and “exclusive or”? How is each of them symbolized? When is a disjunction true? When should parentheses be used in symbolizing compound propositions?

## **Lesson 3: Truth Tables for Determining Truth Values**

How many rows are needed to express all combinations of true and false for two variables? for three variables? for  $n$  variables? What is the general method for determining the truth values of a compound proposition? How does this method differ for propositions using constants with known truth values?

## **Lesson 4: The Conditional**

What type of sentence does the conditional represent? Which part of a conditional is the antecedent? Which part is the consequent? What is the symbol for conditional? When is

a conditional considered false? Are conditionals in English with false antecedents actually truth functional? What other compound proposition is by definition equivalent to the conditional? What are several different ways of expressing conditionals in English? What is the rule of transposition?

### **Lesson 5: The Biconditional**

What type of sentence does the biconditional represent? What is the symbol for biconditional? What other compound proposition is equivalent to the biconditional? When is a biconditional true, and when is it false?

### **Lesson 6: Logical Equivalence**

When are two propositions logically equivalent? What is a tautology? What is a self-contradiction? How is the biconditional used to determine if two propositions are logically equivalent? How is it used to determine if two propositions are contradictory?

### **Lesson 7: Truth Tables for Determining Validity**

What is a valid argument? How can a truth table be used to show that an argument is invalid? How can a truth table be used to show that an argument is valid? Can an invalid argument ever have true premises and a true conclusion?

### **Lesson 8: Shorter Truth Tables for Determining Validity**

What should be initially assumed about an argument when using a shorter truth table to determine the argument's validity? Explain the procedure for determining validity using a shorter truth table.

### **Lesson 9: Using Assumed Truth Values in Shorter Truth Tables**

Can all propositional arguments be analyzed for validity using a shorter truth table of only one line? What must be done when a truth table has no "forced" truth values? If a contradiction appears when a truth value is guessed while using a shorter truth table, what must then be done? Why?

### **Lesson 10: Shorter Truth Tables for Consistency**

What does it mean that a set of propositions are consistent? How can a shorter truth table be used to determine the consistency of a set of propositions?



### **Lesson 11: Shorter Truth Tables for Equivalence**

What does it mean that two propositions are equivalent? What is the method for using a shorter truth table to determine the equivalence of a pair of propositions? How is this similar to using truth tables to determine validity?

### **Lesson 12: The Dilemma**

What is a dilemma? How is a standard constructive dilemma symbolized? How does a destructive dilemma differ from a constructive dilemma? What are the three methods for escaping the horns of a dilemma? Is it possible to use all three methods on every dilemma?



## UNIT 1 REVIEW

# REVIEW EXERCISES

*Students may do these exercises for further review of this unit.*

## ADDITIONAL EXERCISES FOR LESSON 1

1. Are simple propositions truth-functional? Why or why not?

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2. Is a tautology a truth-functional proposition? Is a self-contradiction truth-functional? Explain your answers.

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3. Note which propositions are simple, and which are compound.

Something is rotten in the state of Denmark.	SIMPLE	COMPOUND
If it assume my noble father's person, I'll speak to it.	SIMPLE	COMPOUND
I did love you once.	SIMPLE	COMPOUND
I loved you not.	SIMPLE	COMPOUND
The lady doth protest too much.	SIMPLE	COMPOUND
It is not nor it cannot come to good.	SIMPLE	COMPOUND
Rosencrantz and Guildenstern are dead.	SIMPLE	COMPOUND