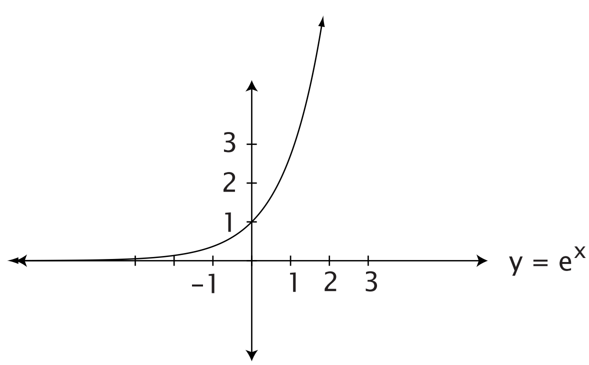
LESSON 6

Exponential and Logarithmic Functions

Exponential functions are of the form $y = a^x$ where *a* is a constant greater than zero and not equal to one and x is a variable. Both $y = 2^x$ and $y = e^x$ are exponential functions. The function, e^x, is extensively used in calculus. You should memorize its approximate value when x = 1. ($e^1 \approx 2.718$)

You should also be able to quickly graph $y = e^x$ without the aid of a calculator. A simple graph of $y = e^x$ is shown below.

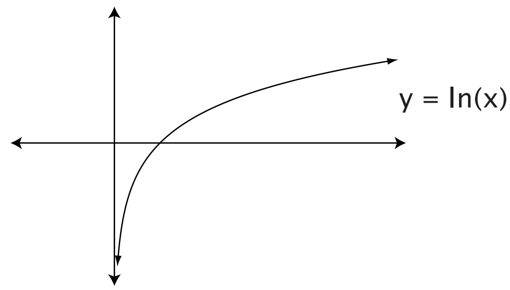
Figure 1



LOGARITHMIC FUNCTIONS

The equation $y = \log_a x$ is the same as $a^y = x$. The inverse of the exponential function is $y = a^x$. In this course we will restrict our study of logarithms to log base e which will be written as ln(x). The equation y = ln(x) is the inverse function of $y = e^x$. Notice that the graph of ln(x) is a reflection of graph of e^x around the line y = x. You should be able to quickly sketch from memory y = ln(x). It will also be important to remember the basic logarithm rules listed on the next page.

Figure 2



I. $\ln(1) = 0$

II. $\ln(e) = 1$

III. $\ln(e^x) = x$

IV. $e^{\ln(x)} = x$

V. Product: $\ln(xy) = \ln(x) + \ln(y)$

VI. Quotient: $\ln(x/y) = \ln(x) - \ln(y)$

VII. Power: $\ln(x^a) = a \ln(x)$

Remember that the natural log of a negative number is undefined. Some books specify $\ln(x)$ as $\ln |x|$. We will use $\ln(x)$ for this book. Be careful to use only positive, non-zero values for x when employing the natural log function.

The natural log function can be used to free variable exponents from their exponential functions. Conversely, the exponential function can do the same for the natural log functions.

Example 1

Solve for x.

$$e^{2x} = 1$$

Taking ln of both sides:

$$\ln\left(e^{2x}\right) = \ln(1) \qquad \text{checking } e^{2(0)} = e^{0} = 1$$

$$2x = 0$$

$$x = 0$$

Example 2

Solve for x.

$$\ln(x + 5) = 0$$

Use each side of the equation as the exponent for e.

$$e^{\ln(x + 5)} = e^{0}$$

$$x + 5 = 1; \text{ so } x = -4$$

Sometimes the equations are complex and we need to use substitution to solve them. See example 3 on the next page.

Example 3
Solve for x.

$$e^{2x} - 4e^{x} + 3 = 0$$

Substituting $u = e^{x}$, $u^2 - 4u + 3 = 0$.

Factoring, we get $(u - 3)(u - 1) = 0$.

Replacing u with e^{x}, we get $(e^{x} - 3)(e^{x} - 1) = 0$.

Solving each factor, we get: $e^{x} = 3$; $e^{x} = 1$.

Taking the ln of both sides:

$$\ln\left(e^{x}\right) = \ln(3) \qquad \ln\left(e^{x}\right) = \ln(1)$$
$$x = \ln(3) \qquad x = 0$$

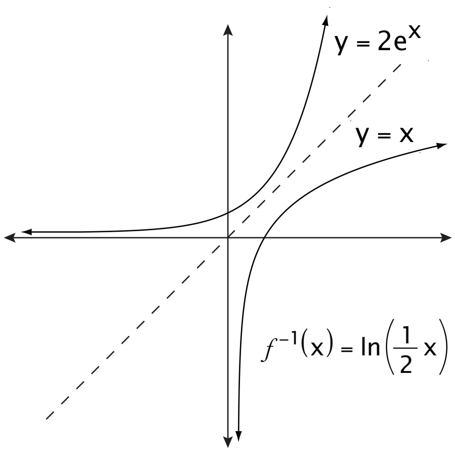
Example 4
Draw the graph of $y = 2e^{x}$ and its inverse.

$y = 2e^{x}$

$x = 2e^{y}$ switch variables

$$\frac{1}{2}x = e^{y}$$
$$\ln\left(\frac{1}{2}x\right) = \ln\left(e^{y}\right)$$
$$\ln\left(\frac{1}{2}x\right) = y$$

$$f^{-1}(x) = \ln\left(\frac{1}{2}x\right)$$



LESSON PRACTICE

Answer the question.

1. Draw the graph of $y = \frac{e^x}{3}$. Find the inverse function. Graph it.

2. Draw the graph of $y = 2e^x$. Find the inverse function. Graph it.

3. Solve for x.

   A. $e^{2x + 1} = 1$

   B. $2e^{3x} = e^0$

   C. $0 = \ln(2x + 5)$

D. $\ln(x) + \ln(5) = 6$

4. Solve for x. (**Hint:** Substitute and factor.)

A. $e^{2x} - 5e^{x} = -6$

B. $2e^{2x} + 7e^{x} = 4$

LESSON PRACTICE

Answer the question.

1. Draw the graph of $y = e^{x + 1}$. Find the inverse function. Graph it.

2. Draw the graph of $y = e^{\frac{x}{2}}$. Find the inverse function. Graph it.

3. Solve for x.

A. $e^{x + \ln(3)} = 2$

B. $e^{x + 1} = e^{2x - 2}$

C. $\ln(x^2 + 3x + 5) = \ln(1 - x)$

D. $\ln\left(\frac{x}{2}\right) = 3$

4. Solve for x.

A. $2\ln^2(x) + 3 = 7\ln(x)$

B. $e^{2x} = 2e^x$

LESSON PRACTICE

Answer the question.

1. Draw the graph of $y = 2x^2$. Find the inverse function. Graph it.

Solve for x.

2. $e^{4x} = e$

3. $\ln(3x - 1) = 1$

4. $e^{2x} - 7e^{x} + 10 = 0$

5. $\ln^{2}(x) = 2\ \ln(x)$

6. $e^{2x} - 3e^{x} + 2 = 0$

6D

LESSON PRACTICE

Solve for x.

1. $e^{2x + 2} = 5$

2. $2e^{2x} + 5e^{x} = 3$

3. $\ln(x + 2) = 2$

4. $\ln(x + 1) + \ln(4) = 3$

5. Solve for x: $\ln(2x - 4) = 2$.

6. Draw the graph of $y = e^{3x}$. Find the inverse function. Graph it.

Circle your answer.

1. Simplify $\frac{\ln(9)}{2} =$

   A. $\ln(4.5)$

   B. $\frac{1}{2}\ln(4.5)$

   C. $\ln(3)$

   D. cannot be simplified

2. Solve for x: $\ln(x) - \ln(4) = 2$.

   A. $\frac{1}{4}e^2$

   B. e^4

   C. e^2

   D. $4e^2$

3. $\ln\left(\frac{6}{3}\right)$ is the same as:

   A. 2
   B. $\ln(18)$
   C. $\ln(3)$
   D. $\ln(6) - \ln(3)$

4. Find the inverse function: $f(x) = \ln(x - 2)$.

   A. $f^{-1}(x) = \ln(x + 2)$
   B. $f^{-1}(x) = e^x + 2$
   C. $f^{-1}(x) = 2e^x - 2$
   D. $f^{-1}(x) = 2e^x$

5. Simplify $\ln\left(\sqrt{2}\right) + \ln\left(\sqrt{10}\right)$.

   A. $\ln\left(\sqrt{12}\right)$
   B. $\sqrt{20}$
   C. $\ln\left(2\sqrt{5}\right)$
   D. cannot be simplified

6. Solve for x: $\ln^2(x) - 5\ln(x) = -4$

A. $x = e, e^4$
B. $x = \ln(4), \ln(5)$
C. $x = e^5, e$
D. $x = \ln(4), e$

7. e^X and $\ln(x)$ are inverse functions. The graph of $y = \ln(x)$ is the reflection of the graph of $y = e^x$ around the:

A. x-axis
B. y-axis
C. origin
D. line $y = x$

8. Solve for x: $e^{2x} = 3e^x$

A. $x = e^x - 3$
B. $x = \ln(3)$
C. $x = \ln(3)$ and $x = 0$
D. $x = 0$

9. Solve for x: $\ln(2) + \ln(x) = 7$

A. $x = \frac{e^2}{7}$

B. $x = 7e^2$

C. $x = 2e^7$

D. $x = \frac{e^7}{2}$

10. Solve for x: $e^{3x-1} = 1$

A. $\frac{1}{3}$

B. $-\frac{1}{3}$

C. 3

D. e^3

7. $\cos(2x) = 0$ for x in $[0, \pi]$

$\cos(\theta) = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ etc

$2x = \frac{\pi}{2}$ $\quad 2x = \frac{3\pi}{2}$ $\quad 2x = \frac{5\pi}{2}$

$x = \frac{\pi}{4}$ $\quad x = \frac{3\pi}{4}$ $\quad x = \frac{5\pi}{4}$

$x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$

8. $\tan\left(\frac{1}{2}x\right) = 0 \left[0, \frac{\pi}{2}\right]$

$\tan(\theta) = 0$ when $\theta = 0, \pi, 2\pi, 3\pi$ etc.

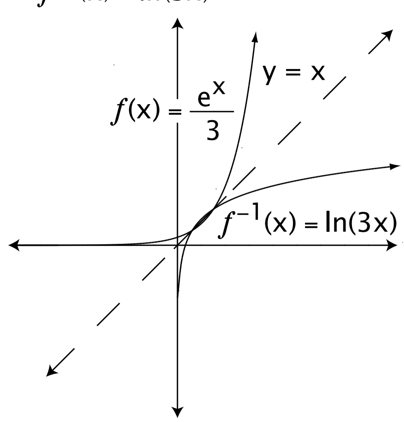
$\frac{1}{2}x = 0$ $\quad \frac{1}{2}x = \pi$

$x = 0$ $\quad x = 2\pi$

$x = 0$ is the only answer in $\left[0, \frac{\pi}{2}\right]$

Lesson Practice 6A

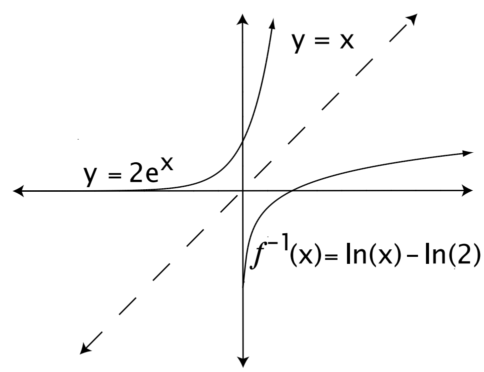
1.
$$y = \frac{e^x}{3}$$
$$x = \frac{e^y}{3} \text{ (Switch variables)}$$
$$3x = e^y$$
$$\ln(3x) = \ln(e^y)$$
$$\ln(3x) = y$$
$$f^{-1}(x) = \ln(3x)$$



2.
$$y = 2e^x$$
$$x = 2e^y \text{ (reverse variables)}$$
$$\ln(x) = \ln(2e^y)$$
$$\ln(x) = \ln 2 + y$$
$$\ln(x) - \ln(2) = y$$
$$f^{-1}(x) = \ln(x) - \ln(2)$$

Note: This problem is the same as example 4 in the instruction manual but solved differently. Both solutions are correct.

$$\ln(x) - \ln(2) = \ln\left(\frac{x}{2}\right)$$



3. A.
$$e^{2x+1} = 1$$
$$\ln(e^{2x+1}) = \ln(1)$$
$$2x + 1 = 0$$
$$2x = -1$$
$$x = -\frac{1}{2}$$

B.
$$2e^{3x} = e^0$$
$$\ln(2e^{3x}) = \ln(e^0)$$
$$\ln(2) + 3x = 0$$
$$3x = -\ln(2)$$
$$x = -\frac{\ln(2)}{3}$$